Exercise Problem Sets 3

Sept. 29. 2023

Problem 1. Let I be an open interval in $\mathbb{R}$, $c \in I$, and $f : I \to \mathbb{R}$ be a function. Show that f is continuous at c if and only if $\lim_{h \to 0} f(c + h) = f(c)$.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(a + b) = f(a)f(b)$ for all $a, b \in \mathbb{R}$.

1. Show that $f(x) \geqslant 0$ for all $x \in \mathbb{R}$.
2. Show that if f is continuous at 0, then f is continuous on $\mathbb{R}$ (that is, f is continuous at every point of $\mathbb{R}$).

Problem 3. Let I be an interval in $\mathbb{R}$ and $f, g : I \to \mathbb{R}$ be continuous functions. Show that if $f(x) = g(x)$ for all $x \in \mathbb{Q} \cap I$, then $f(x) = g(x)$ for all $x \in I$.

Problem 4. Let I be an interval, $c \in I$, and $f : I \to \mathbb{R}$ be a continuous function. Show that if $f(c) \neq 0$, there exists $\delta > 0$ such that $f(x)f(c) > 0$ whenever $|x - c| < \delta$ and $x \in I$.

Problem 5. Construct a function $f : \mathbb{R} \to \mathbb{R}$ so that f is continuous at all integers but nowhere else.

Problem 6. Find the following limits:

1. $\lim_{x \to -\infty} (2x + \sqrt{4x^2 + 3x - 2})$.
2. $\lim_{x \to \infty} \left(x - \sqrt[3]{x^3 + 2x - 3}\right)$.
3. $\lim_{x \to \infty} \frac{[\![x]\!]}{x}$, where $[\![\cdot]\!]$ is the floor function.

Problem 7. True or False: Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. If $|f|$ is continuous at c, so is f.
2. Let I be an interval and $f : I \to \mathbb{R}$ be a continuous function. If $f(x) \neq 0$ for all $x \in I$, then f never change signs; that is, either $f(x) > 0$ for all $x \in I$ or $f(x) < 0$ for all $x \in I$.