Extra Exercise Problem Sets 2

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Problem 1. The second Taylor polynomial for a twice-differentiable function f at $x = c$ is called the quadratic approximation of f at $x = c$. Find the quadratic approximate of the following functions at $x = 0$.

(1) $f(x) = \ln \cos x$ (2) $f(x) = e^{\sin x}$ (3) $f(x) = \tan x$ (4) $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

(5) $f(x) = e^x \sin^2 x$ (6) $f(x) = e^x \ln(1+x)$ (7) $f(x) = (\arctan x)^2$

Problem 2. Let f have derivatives through order n at $x = c$. Show that the n-th Taylor polynomial for f at c and its first n derivatives have the same values that f and its first n derivatives have at $x = c$.

Problem 3. Complete the following.

(1) Let $f, g : [a, b] \to \mathbb{R}$ be continuous and g is sign-definite; that is, $g(x) \geqslant 0$ for all $x \in [a, b]$ or $g(x) \leqslant 0$ for all $x \in [a, b]$. Show that there exists $c \in [a, b]$ such that

$$f(c) \int_a^b g(x)\,dx = \int_a^b f(x)g(x)\,dx. \tag{0.1}$$

(2) Let $f : [a, b] \to \mathbb{R}$ be a function, and $c \in [a, b]$. Prove (by induction) that if f is $(n+1)$-times continuously differentiable on $[a, b]$, then for all $x \in [a, b]$,

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &\quad + (-1)^n \int_c^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\,dt \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^n \int_c^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\,dt. \end{aligned}$$

(3) Use (0.1) to show that if f is $(n+1)$-times continuously differentiable on $[a, b]$ and $c \in [a, b]$, then for all $x \in [a, b]$ there exists a point ξ between x and c such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}.$$

(4) Find and explain the difference between the conclusion above and Taylor's Theorem.

Problem 4. Suppose that f is differentiable on an interval centered at $x = c$ and that $g(x) = b_0 + b_1(x-c) + \cdots + b_n(x-c)^n$ is a polynomial of degree n with constant coefficients $b_0, b_1, \cdots, b_n$. Let $E(x) = f(x) - g(x)$. Show that if we impose on g the conditions

1. $E(c) = 0$ (which means "the approximation error is zero at $x = c$");

2. $\lim_{x\to c} \dfrac{E(x)}{(x-c)^n} = 0$ (which means "the error is negligible when compared to $(x-c)^n$),

then g is the n-th Taylor polynomial for f at c. **Thus, the Taylor polynomial P_n is the only polynomial of degree less than or equal to n whose error is both zero at $x = c$ and negligible when compared with $(x-c)^n$.**

Problem 5. Show that if p is an polynomial of degree n, then

$$p(x+1) = \sum_{k=0}^{n} \frac{p^{(k)}(x)}{k!}.$$

Problem 6. In class we briefly talked about Newton's method for approximating a root/zero r of the equation $f(x) = 0$, and from an initial approximation x_1 we obtained successive approximations x_2, x_3, $\cdots$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \forall\, n \geqslant 1.$$

Show that if f'' exists on an interval I containing r, x_n, and x_{n+1}, and $|f''(x)| \leqslant M$ and $|f'(x)| \geqslant K$ for all $x \in I$, then

$$|x_{n+1} - r| \leqslant \frac{M}{2K}|x_n - r|^2$$

This means that if x_n is accurate to d decimal places, then x_{n+1} is accurate to about $2d$ decimal places. More precisely, if the error at stage n is at most 10^{-m}, then the error at stage $n+1$ is at most $\frac{M}{2K}10^{-2m}$.

Hint: Apply Taylor's Theorem to write $f(r) = P_2(r) + R_2(r)$, where P_2 is the second Taylor polynomial for f at x_n.

Problem 7. Consider a function f with continuous first and second derivatives at $x = c$. Prove that if f has a relative maximum at $x = c$, then the second Taylor polynomial centered at $x = c$ also has a relative maximum at $x = c$.

Problem 8. Let $f : (a, b) \to \mathbb{R}$ be $(n+1)$-times differentiable, and $c \in (a, b)$. In this problem you are ask to derive the remaind associated with the n-th Taylor polynomial for f at c in Schlomilch-Roche form:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!p}(x-c)^p(x-\xi)^{n+1-p}. \tag{0.2}$$

Suppose that $f : (a, b) \to \mathbb{R}$ is $(n+1)$-times differentiable. For a fixed $x \in (a, b)$, define

$$\varphi(z) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(x-z)^k.$$

Note that $\varphi(c) = R_n(x)$. Complete the following.

1. Show that $\varphi'(z) = -\dfrac{f^{(n+1)}(z)}{n!}(x-z)^n$.

2. Apply the Cauchy mean value theorem to the two functions $\varphi(z)$ and $\psi(z) \equiv (x-z)^p$ for some $p \in \{1, 2, \cdots, n\}$; that is,

$$\frac{\varphi(x) - \varphi(c)}{\psi(x) - \psi(c)} = \frac{\varphi'(\xi)}{\psi'(\xi)} \qquad \text{for some } \xi \text{ between } c \text{ and } x,$$

to show (0.2).

3. Use (0.2) to show that

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \qquad \forall\, x \in (-1, 1]. \tag{??}$$

Remark: The remainder in Schlomilch-Roche form with $p = 1$ is called Cauchy remainder, and Lagrange remainder is obtained by letting $p = n+1$ in (0.2).

Problem 9. Suppose that $f : [a, b] \to \mathbb{R}$ is three times continuously differentiable, $h = \dfrac{b-a}{2}$ and $c = \dfrac{a+b}{2}$. Show that there exists $\xi \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{2h} - \frac{h^2}{6} f^{(3)}(\xi).$$

Hint: Find the difference $f(b) - f(a)$ by write f as the sum of its third Taylor polynomial about c and the corresponding remainder. Apply the Intermediate Value Theorem to deal with the sum of the remainders. We note that the identity above implies that

$$\Big| f'(c) - \frac{f(c+h) - f(c-h)}{2h} \Big| \leqslant \frac{h^2}{6} \max_{x \in [c-h, c+h]} \big| f^{(3)}(x) \big|.$$

Problem 10. Suppose that $f : [a, b] \to \mathbb{R}$ is four times continuously differentiable, $h = \dfrac{b-a}{2}$ and $c = \dfrac{a+b}{2}$. Show that there exists $\xi \in (a, b)$ such that

$$f''(c) = \frac{f(a) - 2f(c) + f(b)}{h^2} - \frac{f^{(4)}(\xi)}{12} h^2. \tag{0.3}$$

Hint: Find the sum $f(a) + f(b)$ by write f as the sum of its third Taylor polynomial about c and the corresponding remainder. Apply the Intermediate Value Theorem to deal with the sum of the remainders. We note that the identity above implies that

$$\Big| f''(c) - \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} \Big| \leqslant \frac{h^2}{12} \max_{x \in [c-h, c+h]} \big| f^{(4)}(x) \big|.$$

Problem 11. Suppose that $f : [a, b] \to \mathbb{R}$ is four times continuously differentiable. Show that

$$\Big| \int_a^b f(x)\, dx - \frac{b-a}{6} \big[f(a) + 4f\big(\frac{a+b}{2}\big) + f(b) \big] \Big| \leqslant \frac{2h^5}{45} \max_{x \in [a,b]} \big| f^{(4)}(x) \big| \tag{0.4}$$

through the following steps.

1. Let $c = \dfrac{a+b}{2}$ and $h = \dfrac{b-a}{2}$. Write f as the sum of its third Taylor polynomial about c and the corresponding remainder and conclude that
$$\int_a^b f(x)\,dx = 2hf(c) + \frac{h^3}{3} f''(c) + \int_a^b R_3(x)\,dx\,.$$
2. Show (by Intermediate Value Theorem) that there exists $\xi \in (a,b)$ such that
$$\int_a^b R_3(x)\,dx = \frac{f^{(4)}(\xi)}{24} \int_a^b (x-c)^4\,dx = \frac{f^{(4)}(\xi)}{60} h^5\,. \tag{0.5}$$
3. Use (0.3) in (0.5) and conclude (0.4).

Problem 12. Find the interval of convergence of the following power series.

(1) $\sum\limits_{n=1}^{\infty} \left(1+\dfrac{1}{n}\right)^n x^n$ (2) $\sum\limits_{n=1}^{\infty} (\ln n)x^n$ (3) $\sum\limits_{n=1}^{\infty} \left(\sqrt{n+1}-\sqrt{n}\right)x^n$ (4) $\sum\limits_{n=1}^{\infty} \left(\dfrac{n}{n+1}\right)^{n^2} x^n$

(5) $\sum\limits_{n=1}^{\infty} \dfrac{n!}{(2n)!}x^n$ (6) $\sum\limits_{n=1}^{\infty} \dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}x^{2n+1}$ (7) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n 3\cdot 7\cdot 11\cdot\cdots\cdot(4n-1)}{4^n}x^n$

(8) $\sum\limits_{n=1}^{\infty} \dfrac{1}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}x^n$ (10) $\sum\limits_{n=1}^{\infty} \dfrac{1}{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}x^n$ (9) $\sum\limits_{n=1}^{\infty} \dfrac{n!}{3\cdot 6\cdot 9\cdot\cdots\cdot(3n)}x^n$

(10) $\sum\limits_{n=1}^{\infty} \dfrac{k(k+1)(k+2)\cdots(k+n-1)}{n!}x^n$, where k is a positive integer;

(11) $\sum\limits_{n=0}^{\infty} \dfrac{(n!)^k}{(kn)!}x^n$, where k is a positive integer; (12) $\sum\limits_{n=2}^{\infty} \dfrac{x^n}{n\ln n}$ (13) $\sum\limits_{n=2}^{\infty} \dfrac{x^n}{n(\ln n)^2}$

(14) $\sum\limits_{n=1}^{\infty} \left[2+(-1)^n\right](x+1)^{n-1}$

Problem 13. The function J_0 defined by
$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$
is called the Bessel function of the first kind of order 0. Find its domain (that is, the interval of convergence).

Problem 14. The function J_1 defined by
$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$
is called the Bessel function of the first kind of order 1. Find its domain (that is, the interval of convergence).

Problem 15. The function A defined by
$$A(x) = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots$$
is called an Airy function after the English mathematician and astronomer Sir George Airy (1801–1892). Find the domain of the Airy function.

Problem 16. A function f is defined by

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots ;$$

that is, its coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \geqslant 0$. Find the interval of convergence of the series and find an explicit formula for $f(x)$.

Problem 17. Let $f : (-r, r) \to \mathbb{R}$ be n-times differentiable at 0, and $P_n(x)$ be the n-th Maclaurin polynomial for f.

1. Show that if $g(x) = x^\ell f(x^m)$ for some positive integers m and ℓ, then the $(mn+\ell)$-th Maclaurin polynomial for g is $x^\ell P_n(x^m)$.
2. Show that if $h(x) = x^\ell f(-x^m)$ for some positive integers m and ℓ, then the $(mn + \ell)$-th Maclaurin polynomial for h is $x^\ell P_n(-x^m)$.
3. Find the Maclaurin series for the following functions:

$$(1)\ y = \frac{1}{1+x^2} \quad (2)\ y = x^2 \arctan(x^3) \quad (3)\ y = \ln(1 + x^4) \quad (4)\ y = x \sin(x^3) \cos(x^3).$$

Hint for (1) and (2): See Exercise 3 Problem 4.

Problem 18. To find the sum of the series $\sum\limits_{n=1}^{\infty} \frac{n^2}{2^n}$, express $\frac{1}{1-x}$ as a geometric series, differentiate both sides of the resulting equation with respect to x, multiply both sides of the result by x, differentiate again, multiply by x again, and set x equal to $\frac{1}{2}$. What do you get?

Problem 19. Complete the following.

(1) Use the power series of $y = \arctan x$ to show that

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

(2) Using $x^3 + 1 = (x + 1)(x^2 - x + 1)$, rewrite the integral $\int_0^{\frac{1}{2}} \frac{dx}{x^2 - x + 1}$ and then express $\frac{1}{1+x^3}$ as the sum of a power series to prove the following formula for π:

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \Big(\frac{2}{3n+1} + \frac{1}{3n+2} \Big).$$

Problem 20. Show that the Bessel function of the first kind of order 0, denoted by J_0 and defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2},$$

satisfies the differential equation

$$x^2 y''(x) + x y'(x) + x^2 y(x) = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

Problem 21. Find the power series solution $y(x) = \sum_{k=0}^{\infty} a_k x^k$ to the differential equation

$$y''(x) + y(x) = x, \qquad y(0) = 0, \quad y'(0) = 2.$$

Problem 22. Show that the Bessel function of the first kind of order 1, denoted by J_1 and defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}},$$

satisfies the differential equation

$$x^2 y''(x) + xy'(x) + (x^2 - 1)y(x) = 0, \qquad y(0) = 0, \; y'(0) = \frac{1}{2}.$$

Problem 23. Suppose that $x_1(t)$ and $x_2(t)$ are functions of t satisfying the following equations

$$\begin{aligned} x_1''(t) - x_1(t) &= 0, \qquad x_1(0) = 1, \quad x_1'(0) = 0, \\ x_2''(t) - x_2(t) &= 0, \qquad x_2(0) = 0, \quad x_2'(0) = 1, \end{aligned}$$

where $'$ denotes the derivatives with respect to t.

1. Assume that the function $x_1(t)$ and $x_2(t)$ can be written as a power series (on a certain interval), that is, $x_1(t) = \sum_{k=0}^{\infty} a_k t^k$ and $x_2(t) = \sum_{k=0}^{\infty} b_k t^k$. Show that

$$(k+2)(k+1)a_{k+2} = a_k \quad \text{and} \quad (k+2)(k+1)b_{k+2} = b_k \qquad \forall\, k \geqslant 0.$$

2. Find a_k and b_k, and conclude that x_1 and x_2 are some functions that we have seen before.

3. Find a function $x(t)$ satisfying

$$x''(t) - x(t) = 0, \qquad x(0) = a, \quad x'(0) = b.$$

   Note that x can be written as the linear combination of x_1 and x_2.

Problem 24. Find the series solution to the differential equation

$$y''(x) + x^2 y(x) = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

What is the radius of convergence of this series solution?

Problem 25. In this problem we try to establish the following theorem

Theorem 0.1. *Let the radius of convergence of the power series $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ be r for some $r > 0$.*

1. *If $\sum_{k=0}^{\infty} a_k r^k$ converges, then f is continuous at $c+r$; that is $\lim_{x \to (c+r)^-} f(x) = f(c+r)$.*

2. *If* $\sum_{k=0}^{\infty} a_k(-r)^k$ *converges, then* f *is continuous at* $c-r$*; that is,* $\lim_{x\to(c-r)^+} f(x) = f(c-r)$.

Prove case 1 of the theorem above through the following steps.

(1) Let $A = \sum_{k=0}^{\infty} a_k r^k$, and define

$$g(x) = f(rx+c) - A = -\sum_{k=1}^{\infty} a_k r^k + \sum_{k=1}^{\infty} a_k r^k x^k = \sum_{k=0}^{\infty} b_k x^k,$$

where $b_k = a_k r^k$ for each $k \in \mathbb{N}$ and $b_0 = -\sum_{k=1}^{\infty} a_k r^k$. Show that the radius of convergence of g is 1 and $\sum_{k=0}^{\infty} b_k = 0$. Moreover, show that f is continuous at $c+r$ if and only if g is continuous at 1.

(2) Let $s_n = b_0 + b_1 + \cdots + b_n$ and $S_n(x) = b_0 + b_1 x + \cdots + b_n x^n$. Show that

$$S_n(x) = (1-x)(s_0 + s_1 x + \cdots + s_{n-1}x^{n-1}) + s_n x^n$$

and conclude that

$$g(x) = \lim_{n\to\infty} S_n(x) = (1-x)\sum_{k=0}^{\infty} s_k x^k. \tag{0.6}$$

(3) Use (0.6) to show that g is continuous at 1. Note that you might need to use ε-δ argument.

Problem 26. Complete the following.

1. Find the n-th Maclaurin polynomial $P_n(x)$ of the function $f(x) = \dfrac{1}{1+x}$ and express the remainder term $R_n(x) = f(x) - P_n(x)$ in Lagrange form.

2. Show that if $|x| \leqslant \frac{1}{2}$, then

$$\left|\ln(1+x) - \left(x - \frac{x^2}{2}\right)\right| \leqslant \frac{8}{3}|x|^3$$

by first showing that $|R_1(x)| \leqslant 8x^2$ for $|x| \leqslant \frac{1}{2}$ and then integrating the identity $f(t) = P_1(t) + R_1(t)$ from 0 to x to obtain that

$$\left|\int_0^x R_1(t)\,dt\right| \leqslant \frac{8}{3}|x|^3 \qquad \forall\, x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

3. (Use part 2 to) Justify whether the series $\sum_{k=1}^{\infty} \ln\left(1 + \frac{(-1)^k}{\sqrt{k}}\right)$ converges or not.

4. From part 1 we have

$$\frac{1}{1+x^2} = P_n(x^2) + R_n(x^2).$$

Integrate the identity above on the interval $\left[0, \frac{1}{2}\right]$ and find n such that

$$\left|\arctan\frac{1}{2} - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)2^{2k+1}}\right| < 10^{-8}.$$