Extra Exercise Problem Sets 3

Apr. 28. 2025

Problem 1. Let C be a curve parameterized by the vector-valued function $\boldsymbol{r} : [0,1] \to \mathbb{R}^2$,

$$\boldsymbol{r}(t) = \Big(\frac{e^t - e^{-t}}{e^t + e^{-t}}, \frac{2}{e^t + e^{-t}}\Big), \quad 0 \leqslant t \leqslant 1.$$

(1) Show that C is part of the unit circle centered at the origin.

(2) Plot the curve C. (The plot does not have to be very precise. You only need to specify the starting and end points as well as the orientation.)

(3) Find the length of the curve C.

Problem 2. Let C be the curve given by the parametric equations

$$x(t) = \frac{3 + t^2}{1 + t^2}, \qquad y(t) = \frac{2t}{1 + t^2}$$

on the interval $t \in [0,1]$.

(1) In fact C is the graph of a function $y = f(x)$. Find f.

(2) Find the arc length of the curve C.

(3) Find the area of the surface formed by revolving the curve C about the y-axis.

Problem 3. In class we talked about how to find the total distance that you travel when you walk along a path according to the position vector $\boldsymbol{r} : [a,b] \to \mathbb{R}^2$. The total distance travelled can be computed by

$$\int_a^b \|\boldsymbol{r}'(t)\|\, dt$$

when $\boldsymbol{r}$ is continuously differentiable. Complete the following.

1. Let $\boldsymbol{r} : [0, 4\pi] \to \mathbb{R}^2$ be given by $\boldsymbol{r}(t) = 3\cos t\boldsymbol{i} + 3\sin t\boldsymbol{j}$. Find the image of $[0, 4\pi]$ under $\boldsymbol{r}$.
2. Compute the integral $\displaystyle\int_0^{4\pi} \|\boldsymbol{r}'(t)\|\, dt$. Does it agree with the length of the curve $C \equiv \boldsymbol{r}\big([0, 4\pi]\big)$?

Problem 4. To illustrate that the length of a smooth space curve does not depend on the parametrization you use to compute it, calculate the length of one turn of the helix in Example 1 with the following parametrizations.

1. $\boldsymbol{r}(t) = \cos(4t)\boldsymbol{i} + \sin(4t)\boldsymbol{j} + 4t\boldsymbol{k}$, $t \in \big[0, \frac{\pi}{2}\big]$.
2. $\boldsymbol{r}(t) = \cos\frac{t}{2}\boldsymbol{i} + \sin\frac{t}{2}\boldsymbol{j} + \frac{t}{2}\boldsymbol{k}$, $t \in [0, 4\pi]$.
3. $\boldsymbol{r}(t) = \cos t\boldsymbol{i} - \sin t\boldsymbol{j} - t\boldsymbol{k}$, $t \in [-2\pi, 0]$.

Problem 5. Parametrize the curve

$$\mathbf{r} = \mathbf{r}(t) = \arctan\frac{t}{\sqrt{1-t^2}}\boldsymbol{i} + \arcsin t\boldsymbol{j} + \arccos t\boldsymbol{k}, \quad t \in \Big[-1, 0.5\Big],$$

in the same orientation in terms of arc-length measured from the point where $t = 0$.

Problem 6. Parametrize the curve

$$\mathbf{r} = \mathbf{r}(t) = \arcsin\frac{t}{\sqrt{1+t^2}}\boldsymbol{i} + \arctan t\boldsymbol{j} + \arccos\frac{1}{\sqrt{1+t^2}}\boldsymbol{k}, \quad t \in [-1, 1],$$

in the same orientation in terms of arc-length measured from the point where $t = 0$.

Problem 7. Let C_1 be the polar graph of the polar function $r = 1 + \cos\theta$ (which is a cardioid), and C_2 be the polar graph of the polar function $r = 3\cos\theta$ (which is a circle). See the following figure for reference.

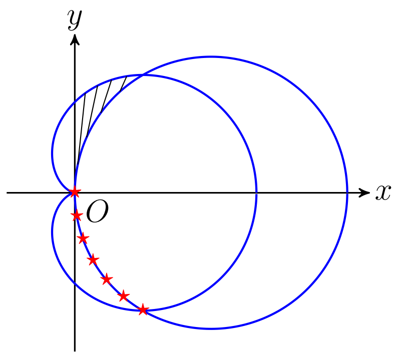

Figure 1: The polar graphs of the polar equations $r = 1 + \cos\theta$ and $r = 3\cos\theta$

(1) Find the intersection points of C_1 and C_2.

(2) Find the line L passing through the lowest intersection point and tangent to the curve C_2.

(3) Identify the curve marked by $\star$ on the θr-plane for $0 \leqslant \theta \leqslant 2\pi$.

(4) Find the area of the shaded region.

Problem 8. Let R be the region bounded by the lemniscate $r^2 = 2\cos 2\theta$ and is outside the circle $r = 1$ (see the shaded region in the graph).

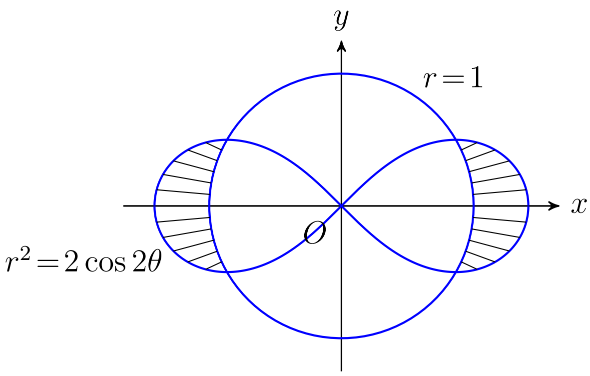

Figure 2: The polar graphs of the polar equations $r^2 = 2\cos 2\theta$ and $r = 1$

(1) Find the area of R.

(2) Find the slope of the tangent line passing thought the point on the lemniscate corresponding to $\theta = \dfrac{\pi}{6}$.

(3) Find the volume of the solid of revolution obtained by rotating R about the x-axis by complete the following:

(a) Suppose that (x, y) is on the lemniscate. Then (x, y) satisfies

$$y^4 + a(x)y^2 + b(x) = 0 \tag{0.1}$$

for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$.

(b) Solving (0.1), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1+4x^2}$ for some constants c_1, c_2 and c_3. Then the volume of interests can be computed by

$$I = 2 \times \left[\pi \int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} c(x)dx - \pi \int_{\frac{\sqrt{3}}{2}}^{1} d(x)dx\right].$$

Compute $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{1} \left[d(x) - (1-x^2)\right]dx$.

(c) Evaluate I by first computing the integral $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} \sqrt{1+4x^2}\,dx$, and then find I.

(4) Find the surface area of the surface of revolution obtained by rotating the boundary of R about the x-axis.

Problem 9. Let R be the region bounded by the circle $r = 1$ and outside the lemniscate $r^2 = -2\cos 2\theta$, and is located on the right half plane (see the shaded region in the graph).

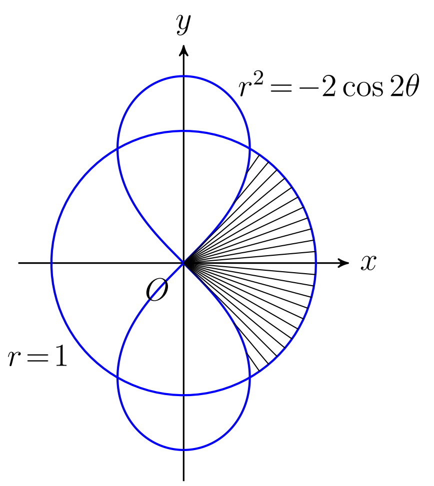

Figure 3: The polar graphs of the polar equations $r = 1$ and $r^2 = -2\cos 2\theta$

(1) Find the points of intersection of the circle $r = 1$ and the lemniscate $r^2 = -2\cos 2\theta$.

(2) Show that the straight line $x = \frac{1}{2}$ is tangent to the lemniscate at the points of intersection on the right half plane.

(3) Find the area of R.

(4) Find the volume of the solid of revolution obtained by rotating R about the x-axis by complete the following:

(a) Suppose that (x, y) is on the lemniscate. Then (x, y) satisfies

$$y^4 + a(x)y^2 + b(x) = 0 \tag{0.2}$$

for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$.

(b) Solving (0.2), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1-4x^2}$ for some constants c_1, c_2 and c_3. Then the volume of interests can be computed by

$$I = \pi \int_0^{\frac{1}{2}} c(x)dx + \pi \int_{\frac{1}{2}}^1 d(x)dx.$$

Compute $\int_{\frac{1}{2}}^1 \Big[d(x) - (1-x^2)\Big] dx$.

(c) Evaluate I by first computing the integral $\int_0^{\frac{1}{2}} \sqrt{1-4x^2}dx$, and then find I.

(5) Find the area of the surface of revolution obtained by rotating the boundary of R about the x-axis.

Problem 10. Let C_1, C_2 be the curves given by polar coordinate $r = 1 - 2\sin\theta$ and $r = 4 + 4\sin\theta$, respectively, and the graphs of C_1 and C_2 are given in Figure 4.

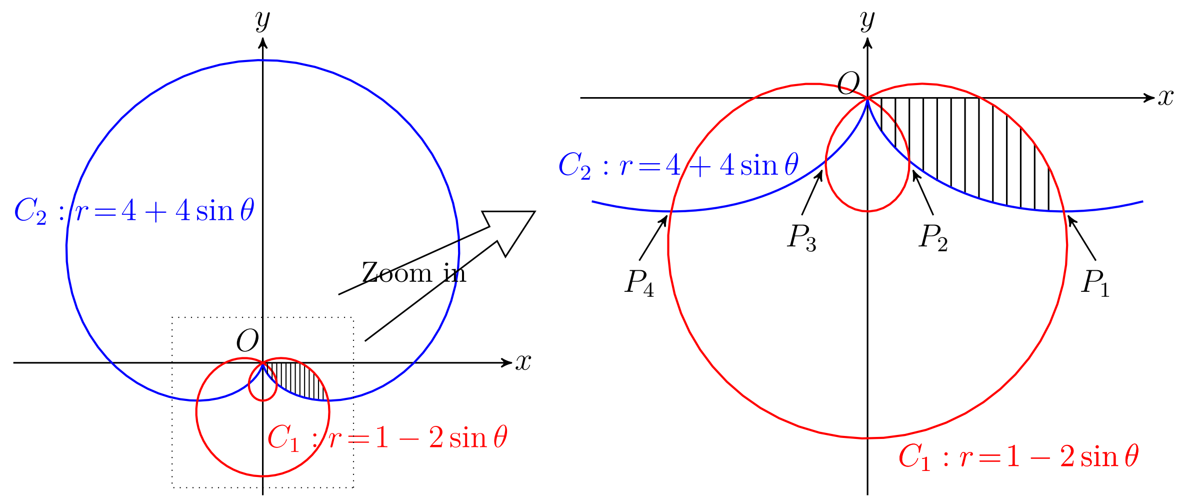

Figure 4: The polar graphs of the polar equations $r = 1 - 2\sin\theta$ and $r = 4 + 4\sin\theta$

(1) Let $P_1, \cdots, P_4$ be four points of intersection of curves C_1 and C_2 as shown in Figure 4 (the fifth one is the origin). What are the Cartesian coordinates of P_1 and P_2?

(2) Let L_1 and L_2 be two straight lines passing P_1 and tangent to C_1, C_2, respectively. Find the cosine value of the acute/smaller angle between L_1 and L_2.

(3) Compute the area of the shaded region.