Extra Exercise Problem Sets 4

Apr. 28. 2025

Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

$$f(x,y) + f(y,z) + f(z,x) = 0 \qquad \forall\, x,y,z \in \mathbb{R}.$$

Show that there exists $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x,y) = g(x) - g(y) \qquad \forall\, x,y \in \mathbb{R}.$$

Problem 2. In the following sub-problems, find the limit if it exists or explain why it does not exist.

(1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y}$ (2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2}$ (3) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

(4) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ (5) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ (6) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

(7) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$ (8) $\displaystyle\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}$ (9) $\displaystyle\lim_{(x,y)\to(0,0)} x\cos\frac{1}{y}$

(10) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$ (11) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+zx}{x^2+y^2+z^2}$

(12) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$ 13) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \arctan\frac{1}{x^2+y^2+z^2}$

Problem 3. Discuss the continuity of the functions given below.

1. $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy} & \text{if } xy \neq 0, \\ 1 & \text{if } xy = 0. \end{cases}$

2. $f(x,y) = \begin{cases} \dfrac{e^{-x^2-y^2}-1}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 1 & \text{if } (x,y) = (0,0). \end{cases}$

3. $f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Problem 4. Let $f(x,y) = \begin{cases} 0 & \text{if } y \leqslant 0 \text{ or } y \geqslant x^4, \\ 1 & \text{if } 0 < y < x^4. \end{cases}$

1. Show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along any path through $(0,0)$ of the form $y = mx^\alpha$ with $0 < \alpha < 4$.

2. Show that f is discontinuous on two entire curves.

Problem 5. Find $\displaystyle \frac{\partial}{\partial x}\bigg|_{(x,y,z)=(\ln 4,\ln 9,2)} \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!z^n}$. Do not write the answer in terms of an infinite sum.

Problem 6. Let $f(x,y) = (x^2+y^2)^{\frac{2}{3}}$. Find the partial derivative $\dfrac{\partial f}{\partial x}$.

Problem 7. Let $f(x,y,z) = xy^2z^3 + \arcsin(x\sqrt{z})$. Find f_{xzy} in the region $\big\{(x,y,z)\,\big|\,|x^2z| < 1\big\}$.

Problem 8. Let $\vec{\boldsymbol{a}} = (a_1, a_2, \cdots, a_n)$ be a unit vector, $\vec{\boldsymbol{x}} = (x_1, x_2, \cdots, x_n)$, and $f(x_1, x_2, \cdots, x_n) = \exp(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{x}})$. Show that

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = f.$$

Problem 9. Let $f(x,y) = x(x^2+y^2)^{-\frac{3}{2}} e^{\sin(x^2y)}$. Find $f_x(1,0)$.

Problem 10. Let $f(x,y) = \displaystyle\int_1^y \frac{dt}{\sqrt{1-x^3t^3}}$. Show that

$$f_x(x,y) = \int_1^y \left(\frac{\partial}{\partial x}\frac{1}{\sqrt{1-x^3t^3}}\right) dt$$

in the region $\big\{(x,y)\,\big|\,x<1, y>1 \text{ and } xy<1\big\}$.

Problem 11. The gas law for a fixed mass m of an ideal gas at absolute temperature T, pressure P, and volume V is $PV = mRT$, where R is the gas constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1.$$

Problem 12. The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Find $\dfrac{\partial R}{\partial R_1}$ by directly taking the partial derivative of the equation above.

Problem 13. Find the value of $\dfrac{\partial z}{\partial x}$ at the point $(1,1,1)$ if the equation

$$xy + z^3x - 2yz = 0$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Problem 14. Find the value of $\dfrac{\partial x}{\partial z}$ at the point $(1,-1,-3)$ if the equation

$$xz + y\ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivative exists.

Problem 15. Define

$$f(x,y) = \begin{cases} x^2 \arctan\dfrac{y}{x} - y^2 \arctan\dfrac{x}{y} & \text{if } x, y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

Problem 16. Show that each of the following functions is not differentiable at the origin.

(1) $f(x, y) = \sqrt[3]{x} \cos y$ (2) $f(x, y) = \sqrt{|xy|}$

Problem 17. In the following, show that both $f_x(0,0)$ and $f_y(0,0)$ both exist but that f is not differentiable at $(0,0)$.

$$(1)\ f(x, y) = \begin{cases} \dfrac{5x^2y}{x^3+y^3} & \text{if } x^3+y^3 \neq 0, \\ 0 & \text{if } x^3+y^3 = 0. \end{cases}$$

$$(2)\ f(x, y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$$(3)\ f(x, y) = \begin{cases} \dfrac{3x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$$(4)\ f(x, y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Problem 18. Let $f, g : (a, b) \to \mathbb{R}$ be real-valued function, $h(x, y) = f(x)g(y)$, and $c, d \in (a, b)$. Show that if f is differentiable at c and g is differentiable at d, then h is differentiable at (c, d).

Problem 19. Show that the function $f(x, y) = \sqrt{x^2+y^2} \sin \sqrt{x^2+y^2}$ is differentiable at $(0,0)$.

Problem 20. Investigate the differentiability of the following functions at the point $(0,0)$.

$$(1)\ f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \qquad (2)\ f(x, y) = \begin{cases} \dfrac{xy}{x+y^2} & \text{if } x+y^2 \neq 0, \\ 0 & \text{if } x+y^2 = 0 \end{cases}$$

$$(3)\ f(x, y) = \begin{cases} (x^2+y^2) \sin \dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Problem 21. Assume that f is a continuous function of two variable satisfying that

$$\lim_{(x,y)\to(\pi,-1)} \frac{f(x,y) + y^2 \sin x}{(x-\pi)^2 + (y+1)^2} = 0.$$

Note that the equality above does **NOT** imply that $f(x, y) = -y^2 \sin x$.

1. Find $f_x(\pi, -1)$ and $f_y(\pi, -1)$.
2. Prove or disprove that f is differentiable at $(\pi, -1)$.

Problem 22. Use the chain rule for functions of several variables to compute $\dfrac{dz}{dt}$ or $\dfrac{dw}{dt}$.

(1) $z = \sqrt{1+xy}$, $x = \tan t$, $y = \arctan t$.

(2) $w = x\exp\left(\frac{y}{z}\right)$, $x = t^2$, $y = 1-t$, $z = 1+2t$.

(3) $w = \ln\sqrt{x^2+y^2+z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$.

(4) $w = xy\cos z$, $x = t$, $y = t^2$, $z = \arccos t$.

(5) $w = 2ye^x - \ln z$, $x = \ln(t^2+1)$, $y = \arctan t$, $z = e^t$.

Problem 23. Use the chain rule for functions of several variables to compute $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

(1) $z = \arctan(x^2+y^2)$, $x = s\ln t$, $y = te^s$.

(2) $z = \arctan\dfrac{x}{y}$, $x = s\cos t$, $y = s\sin t$.

(3) $z = e^x\cos y$, $x = st$, $y = s^2+t^2$.

Problem 24. Assume that $z = f\left(ts^2, \frac{s}{t}\right)$, $\dfrac{\partial f}{\partial x}(x,y) = xy$, $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{x^2}{2}$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Problem 25. Find the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at given points.

(1) $\sin(x+y)+\sin(y+z)+\sin(x+z) = 0$, $(x,y,z) = (\pi,\pi,\pi)$.

(2) $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$, $(x,y,z) = (1,\ln 2,\ln 3)$.

(3) $z = e^x\cos(y+z)$, $(x,y,z) = (0,-1,1)$.

Problem 26. Let f be differentiable, and $z = \dfrac{1}{y}\big[f(ax+y)+g(ax-y)\big]$. Show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{y^2}\frac{\partial}{\partial y}\left(y^2\frac{\partial z}{\partial y}\right).$$

Problem 27. Suppose that we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a differentiable function $z = f(x,y)$.

(1) Show that $\dfrac{\partial z}{\partial r} = f_x\cos\theta + f_y\sin\theta$ and $\dfrac{1}{r}\dfrac{\partial r}{\partial \theta} = -f_x\sin\theta + f_y\cos\theta$.

(2) Solve the equations in part (1) to express f_x and f_y in terms of $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$.

(3) Show that $(f_x)^2 + (f_y)^2 = \left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2$.

(4) Suppose in addition that f_x and f_y are differentiable. Show that

$$f_{xx} + f_{yy} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}.$$

Problem 28. Let f be a twice continuously differentiable function. Suppose that we substitute cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$ in $w = f(x,y,z)$ to obtain $W = f(r\cos\theta, r\sin\theta, z)$ (so that W is a function of r, θ and z). Show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{\partial^2 W}{\partial z^2}.$$

Problem 29. Let R be an open region in $\mathbb{R}^2$ and $f : R \to \mathbb{R}$ be a real-valued function. In class we have talked about the differentiability of f. For $k \geqslant 2$, the k-times differentiability of f is defined inductively: for $k \in \mathbb{N}$, f is said to be $(k+1)$-times differentiable at (a,b) if the k-th partial derivative $\dfrac{\partial^k f}{\partial x^{k-j}\partial y^j}$ is differentiable at (a,b) for all $0 \leqslant j \leqslant k$ $\Big($note that in order to achieve this, $\dfrac{\partial^k f}{\partial x^{k-j}\partial y^j}$ has to be defined in a neighborhood of (a,b) for all $0 \leqslant j \leqslant k\Big)$. f is said to be k-times differentiable on R if f is k-times differentiable at (a,b) for all $(a,b) \in R$. f is said to be k-times continuously differentiable on R if the k-th partial derivative $\dfrac{\partial^k f}{\partial x^{k-j}\partial y^j}$ is continuous at (a,b) for all $0 \leqslant j \leqslant k$.

(1) Show that if f is $(k+1)$-times differentiable on R, then f is k-times continuously differentiable on R.

(2) Show that if f is k-times continuously differentiable on R, then f is k-times differentiable on R.

Hint: In this problem Theorem **??** is used (without proving yet).

Problem 30. Let $f(x,) = \sqrt[3]{xy}$.

(1) Show that f is continuous at $(0,0)$.

(2) Show that f_x and f_y exist at the origin but that the directional derivatives at the origin in all other directions do not exist.

Problem 31. Let

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(1) Show that the directional derivative of f at the origin exists in all directions $\boldsymbol{u}$, and

$$(D_{\boldsymbol{u}} f)(0,0) = \Big(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\Big) \cdot \boldsymbol{u}.$$

(2) Determine whether f is differentiable at $(0,0)$ or not.

Problem 32. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x,y,z) = \begin{cases} \dfrac{xyz}{x^2 + y^2 + z^2} & \text{if } (x,y,z) \neq (0,0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find the direction (u,v,w) (satisfying $u^2 + v^2 + w^2 = 1$) along which the value of the function f at $(0,0,0)$ increases most rapidly.

Problem 33. Let $\boldsymbol{u} = (a,b)$ be a unit vector and f be twice continuously differentiable. Show that

$$D_{\boldsymbol{u}}^2 f = f_{xx} a^2 + 2 f_{xy} ab + f_{yy} b^2,$$

where $D_{\boldsymbol{u}}^2 f = D_{\boldsymbol{u}}(D_{\boldsymbol{u}} f)$.

Problem 34. Show that the operation of taking the gradient of a function has the given property. Assume that u and v are differentiable functions of x and y and that a, b are constants.

(1) $\nabla(au + bv) = a\nabla u + b\nabla v$. (2) $\nabla(uv) = u\nabla v + v\nabla u$.

(3) $\nabla\big(\frac{u}{v}\big) = \dfrac{v\nabla u - u\nabla v}{v^2}$. (4) $\nabla(u^n) = nu^{n-1}\nabla u$.

Problem 35. Show that the equation of the tangent plane to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1.$$

Problem 36. Show that the equation of the tangent plane to the elliptic paraboloid $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ at the point (x_0, y_0, z_0) can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}.$$

Problem 37. Find all planes that pass through the points $(0, -2, -1)$ and $(2, 1, 1)$ and are tangent to the paraboloid $z = x^2 + y^2$.

Problem 38. Let f be a differentiable function and consider the surface $z = xf\big(\frac{y}{x}\big)$. Show that the tangent plane at any point (x_0, y_0, z_0) on the surface passes through the origin.

Problem 39. Prove that the angle of inclination θ of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) satisfies

$$\cos\theta = \frac{1}{\sqrt{f_x(x_0, y_0)^2 + f_y(x_0, y_0)^2 + 1}}.$$

Problem 40. In the following problems, find all relative extrema and saddle points of the function. Use the Second Partials Test when applicable.

(1) $f(x, y) = x^2 - xy - y^2 - 3x - y$ (2) $f(x, y) = 2xy - \dfrac{1}{2}(x^4 + y^4) + 1$

(3) $f(x, y) = xy - 2x - 2y - x^2 - y^2$ (4) $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

(5) $f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$ (6) $f(x, y) = \dfrac{1}{x} + xy + \dfrac{1}{y}$

(7) $f(x, y) = \ln(x + y) + x^2 - y$ (8) $f(x, y) = 2\ln x + \ln y - 4x - y$

(9) $f(x, y) = xy\exp\big(-\dfrac{x^2 + y^2}{2}\big)$ (10) $f(x, y) = xy + e^{-xy}$

(11) $f(x, y) = (x^2 + y^2)e^{-x}$ (12) $f(x, y) = \Big(\dfrac{1}{2} - x^2 + y^2\Big)\exp(1 - x^2 - y^2)$

Problem 41. In the following problems, find the absolute extrema of the function over the region R (which contains boundaries).

(1) $f(x, y) = x^2 + xy$, and $R = \big\{(x, y)\,\big|\,|x| \leqslant 2, |y| \leqslant 1\big\}$

(2) $f(x, y) = 2x - 2xy + y^2$, and R is the region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 1$.

(3) $f(x, y) = \dfrac{4xy}{(x^2+1)(y^2+1)}$, and $R = \{(x, y) \,|\, 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$.

(4) $f(x, y) = xy^2$, and $R = \{(x, y) \,|\, x \geqslant 0, y \geqslant 0, x^2 + y^2 \leqslant 3\}$.

(5) $f(x, y) = 2x^3 + y^4$, and $R = \{(x, y) \,|\, x^2 + y^2 \leqslant 1\}$.

Problem 42. Show that $f(x, y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that the discriminant $f_{xx}f_{yy} - f_{xy}^2 = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point

Problem 43. Show that $f(x, y) = x^2ye^{-x^2-y^2}$ has maximum values at $\big(\pm 1, \frac{1}{\sqrt{2}}\big)$ and minimum values at $\big(\pm 1, -\frac{1}{\sqrt{2}}\big)$. Show also that f has infinitely many other critical points and the discriminant $f_{xx}f_{yy} - f_{xy}^2 = 0$ at each of them. Which of them give rise to maximum values? Minimum values? Saddle points?

Problem 44. Find two numbers a and b with $a \leqslant b$ such that

$$\int_a^b \sqrt[3]{24 - 2x - x^2}\, dx$$

has its largest value.

Problem 45. Find a straight line such that the sum of the squared distances from the line to the points $(0, 0)$, $(1, 0)$, and $(0, 1)$ is minimized. Apply the second partials test if applicable.

Problem 46. Let $m > n$ be natural numbers, and A be an $m \times n$ real matrix, $\boldsymbol{b} \in \mathbb{R}^m$ be a vector.

(1) Show that if the minimum of the function $f(x_1, \cdots, x_n) = \|A\boldsymbol{x} - \boldsymbol{b}\|$ occurs at the point $\boldsymbol{c} = (c_1, \cdots, c_n)$, then $\boldsymbol{c}$ satisfies $A^{\mathrm{T}}A\boldsymbol{c} = A^{\mathrm{T}}\boldsymbol{b}$.

(2) Find the relation between the linear regression and (1).

Problem 47. Let $\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}$ be n points with $x_i \neq x_j$ if $i \neq j$. Use the Second Partials Test to verify that the formulas for a and b given by

$$a = \frac{n\sum\limits_{i=1}^n x_iy_i - \Big(\sum\limits_{i=1}^n x_i\Big)\Big(\sum\limits_{i=1}^n y_i\Big)}{n\sum\limits_{i=1}^n x_i^2 - \Big(\sum\limits_{i=1}^n x_i\Big)^2} \qquad \text{and} \qquad b = \frac{1}{n}\Big(\sum_{i=1}^n y_i - a\sum_{i=1}^n x_i\Big)$$

indeed minimize the function $S(a, b) = \sum\limits_{i=1}^n (ax_i + b - y_i)^2$.

Problem 48. Let A be a full rank $m \times n$ real matrix, where $m < n$. and A have full rank. For a given $\boldsymbol{b} \in \mathbb{R}^m$, show that the function f given by

$$f(x_1, \cdots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$$

under the constraint $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{x} = [x_1, \cdots, x_n]^{\mathrm{T}}$, attains its minimum at the point $A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}\boldsymbol{b}$.

Problem 49. The Shannon index (sometimes called the Shannon-Wiener index or Shannon-Weaver index) is a measure of diversity in an ecosystem. For the case of three species, it is defined as

$$H = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3,$$

where p_i is the proportion of species i in the ecosystem.

(1) Express H as a function of two variables using the fact that $p_1 + p_2 + p_3 = 1$.

(2) What is the domain of H?

(3) Find the maximum value of H. For what values of p_1, p_2, p_3 does it occur?

Problem 50. Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq,$$

where p, q, and r are the proportions of A, B, and O in the population. Use the fact that $p+q+r = 1$ to show that P is at most $\frac{2}{3}$.

Problem 51. Find an equation of the plane that passes through the point $(1, 2, 3)$ and cuts off the smallest volume in the first octant.

Problem 52. Use the method of Lagrange multipliers to complete the following.

(1) Maximize $f(x, y) = \sqrt{6 - x^2 - y^2}$ subject to the constraint $x + y - 2 = 0$.

(2) Minimize $f(x, y) = 3x^2 - y^2$ subject to the constraint $2x - 2y + 5 = 0$.

(3) Minimize $f(x, y) = x^2 + y^2$ subject to the constraint $xy^2 = 54$.

(4) Maximize $f(x, y, z) = e^{xyz}$ subject to the constraint $2x^2 + y^2 + z^2 = 24$.

(5) Maximize $f(x, y, z) = \ln(x^2+1)+\ln(y^2+1)+\ln(z^2+1)$ subject to the constraint $x^2+y^2+z^2 = 12$.

(6) Maximize $f(x, y, z) = x + y + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

(7) Maximize $f(x, y, z, t) = x + y + z + t$ subject to the constraint $x^2 + y^2 + z^2 + t^2 = 1$.

(8) Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2z = 6$ and $x + y = 12$.

(9) Maximize $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 + z^2 = 36$ and $2x + y - z = 2$.

(10) Maximize $f(x, y, z) = yz + xy$ subject to the constraint $xy = 1$ and $y^2 + z^2 = 1$.

Problem 53. Use the method of Lagrange multipliers to find the extreme value of the function $f(x, y, z) = x^2 + y^2 + (z - 2)^2$ on the set

$$R = \left\{(x, y, z) \,\middle|\, (x^2 + y^2)(1 - x^2 - y^2) \leqslant z \leqslant 1 - x^2 - y^2\right\}.$$

Hint:

1. Note that for $(x, y, z) \in R$, one must have $x^2 + y^2 \leqslant 1$.
2. Suppose that $g(x, y, z) = x^2 + y^2 + z - 1$ and $h(x, y, z) = (x^2 + y^2)(x^2 + y^2 - 1) + z$. When considering the case $g(x, y, z) = h(x, y, z) = 0$, first show that $g(x, y, z) = h(x, y, z) = 0$ if and only if $x^2 + y^2 = 1$. It will be much easier to compute $(\nabla g) \times (\nabla h)$ given $g = h = 0$ based on this fact.
3. You may need the fact that the real root of the equation $2\mu^3 + 7\mu^2 + 1 = 0$ is about -3.5399.

Problem 54. Use the method of Lagrange multipliers to find the extreme values of the function $f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$ subject to the constraint $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$.

Problem 55. (1) Use the method of Lagrange multipliers to show that the product of three positive numbers x, y, and z, whose sum has the constant value S, is a maximum when the three numbers are equal. Use this result to show that

$$\frac{x + y + z}{3} \geqslant \sqrt[3]{xyz} \qquad \forall\, x, y, z > 0.$$

(2) Generalize the result of part (1) to prove that the product $x_1 x_2 x_3 \cdots x_n$ is maximized, under the constraint that $\sum\limits_{i=1}^{n} x_i = S$ and $x_i \geqslant 0$ for all $1 \leqslant i \leqslant n$, when

$$x_1 = x_2 = x_3 = \cdots = x_n.$$

Then prove that

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leqslant \frac{x_1 + x_2 + \cdots + x_n}{n} \qquad \forall\, x_1, x_2, \cdots, x_n \geqslant 0.$$

Problem 56. (1) Maximize $\sum\limits_{i=1}^{n} x_i y_i$ subject to the constraints $\sum\limits_{i=1}^{n} x_i^2 = 1$ and $\sum\limits_{i=1}^{n} y_i^2 = 1$.

(2) Put $x_i = \dfrac{a_i}{\sqrt{\sum\limits_{j=1}^{n} a_j^2}}$ and $y_i = \dfrac{b_i}{\sqrt{\sum\limits_{j=1}^{n} b_j^2}}$ to show that

$$\sum_{i=1}^{n} a_i b_i \leqslant \sqrt{\sum_{j=1}^{n} a_j^2}\sqrt{\sum_{j=1}^{n} b_j^2}$$

for any numbers $a_1, a_2, \cdots, a_n$, $b_1, b_2, \cdots, b_n$. This inequality is known as the Cauchy-Schwarz Inequality.

Problem 57. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

Problem 58. If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is to enclose the circle $x^2 + y^2 = 2y$, what values of a and b minimize the area of the ellipse?

Problem 59. (1) Use the method of Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

(2) Use the method of Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.

Hint: Use Heron's formula for the area:

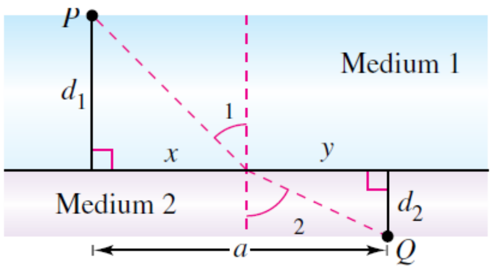
$$A = \sqrt{s(s-x)(s-y)(s-z)},$$

where $s = \frac{p}{2}$ and x, y, z are the lengths of the sides.

Problem 60. When light waves traveling in a transparent medium strike the surface of a second transparent medium, they tend to "bend" in order to follow the path of minimum time. This tendency is called refraction and is described by Snell' s Law of Refraction,

$$\frac{\sin\theta_1}{\mathrm{v}_1} = \frac{\sin\theta_2}{\mathrm{v}_2},$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure, and v_1 and v_2 are the velocities of light in the two media. Use the method of Lagrange multipliers to derive this law using $x + y = a$.



Problem 61. A set $C \subseteq \mathbb{R}^n$ is said to be convex if

$$t\boldsymbol{x} + (1-t)\boldsymbol{y} \in C \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in C \text{ and } t \in [0,1].$$

(一個 $\mathbb{R}^n$ 中的集合 C 被稱為凸集合如果 C 中任兩點 $\boldsymbol{x}$ 與 $\boldsymbol{y}$ 之連線所形成的線段也在 C 中)。

Suppose that $C \subseteq \mathbb{R}^n$ is a convex set, and $f : C \to \mathbb{R}$ be a differentiable real-valued function. Show that if f on C attains its minimum at a point $\boldsymbol{x}^*$, then

$$(\nabla f)(\boldsymbol{x}^*) \cdot (\boldsymbol{x} - \boldsymbol{x}^*) \geqslant 0 \qquad \forall\, \boldsymbol{x} \in C. \tag{$\star$}$$

Hint: Recall that $(\nabla f)(\boldsymbol{x}^*) \cdot (\boldsymbol{x} - \boldsymbol{x}^*)$, when f is differentiable at x^*, is the directional derivative of f at $\boldsymbol{x}^*$ in the "direction" $(\boldsymbol{x} - \boldsymbol{x}^*)$.

Remark: A point $\boldsymbol{x}^*$ satisfying ($\star$) is sometimes called a ***stationary point*** of f in C.

Problem 62. Let B be the unit closed ball centered at the origin given by

$$B = \left\{ \boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \,\middle|\, \|\boldsymbol{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \leqslant 1 \right\},$$

and $f : B \to \mathbb{R}$ be a differentiable real-valued function. Consider the minimization problem $\min\limits_{\boldsymbol{x} \in B} f(\boldsymbol{x})$.

(1) Show that if f attains its minimum at $\boldsymbol{x}^* \in B$, then there exists $\lambda \leqslant 0$ such that

$$(\nabla f)(\boldsymbol{x}^*) = \lambda \boldsymbol{x}^* .$$

(2) Find the minimum of the function $f(x, y) = x^2 + 2y^2 - x$ on the unit closed disk centered at the origin $\left\{(x, y) \,\middle|\, x^2 + y^2 \leqslant 1\right\}$ using (1).

Problem 63. Let $\boldsymbol{a} \in \mathbb{R}^3$ be a vector, $b \in \mathbb{R}$, and C be a half plane given by

$$C = \left\{ \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \,\middle|\, \boldsymbol{a} \cdot \boldsymbol{x} \leqslant b \right\},$$

and $f : C \to \mathbb{R}$ be a differentiable real-valued function. Consider the minimization problem $\min\limits_{\boldsymbol{x} \in C} f(\boldsymbol{x})$. Show that if f attains its minimum at $\boldsymbol{x}^* \in C$, then there exists $\lambda \leqslant 0$ such that

$$(\nabla f)(\boldsymbol{x}^*) = \lambda \boldsymbol{a}.$$