Extra Exercise Problem Sets 5

May. 23. 2025

Problem 1. Evaluate the following iterated integrals.

(1) $\int_{-1}^{1}\Big(\int_0^1 ye^{x^2+y^2}\,dx\Big)dy$ (2) $\int_0^2\Big(\int_y^{\sqrt{8-y^2}}\sqrt{x^2+y^2}\,dx\Big)dy$ (3) $\int_0^1\Big(\int_{\sqrt{y}}^1 e^{x^3}\,dx\Big)dy$

(4) $\int_0^1\Big(\int_y^1 \frac{1}{1+x^4}\,dx\Big)dy$ (5) $\int_0^4\Big(\int_{\frac{x}{2}}^2 \sin(y^2)\,dy\Big)dx$ (6) $\int_0^4\Big(\int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\Big)\,dx$

(7) $\int_0^2\Big(\int_x^2 x\sqrt{1+y^3}\,dy\Big)dx$ (8) $\int_0^2\Big(\int_{\frac{y}{2}}^1 \exp(x^2)\,dx\Big)dy$ (9) $\int_0^1\Big(\int_0^1 \frac{y}{1+x^2y^2}\,dx\Big)\,dy$

(10) $\int_0^{\frac{\pi}{2}}\Big(\int_x^{\frac{\pi}{2}} \frac{\sin y}{y}\,dy\Big)dx$ (11) $\int_0^2\Big(\int_{y^2}^4 \sqrt{x}\sin x\,dx\Big)dy$ (12) $\int_0^2\Big(\int_0^{4-x^2} \frac{xe^{2y}}{4-y}dy\Big)dx$

(13) $\int_0^1\Big(\int_{\arcsin y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\,dx\Big)dy$

Problem 2. Evaluate the double integral $\iint_R f(x,y)\,dA$ with the following f and R.

(1) $f(x,y) = y^2e^{xy}$, and R is the region bounded by $y = x$, $y = 4$ and $x = 0$.

(2) $f(x,y) = xy$, and R is the region bounded by the line $y = x-1$ and parabola $y^2 = 2x+6$.

(3) $f(x,y) = x^2 + x^2y^3 - y^2\sin x$, and $R = \big\{(x,y)\,\big|\,|x|+|y| \leqslant 1\big\}$.

(4) $f(x,y) = |x|+|y|$, and $R = \big\{(x,y)\,\big|\,|x|+|y| \leqslant 1\big\}$.

(5) $f(x,y) = xy$, and R is the region in the first quadrant bounded by curves $x^2+y^2 = 4$, $x^2+y^2 = 9$, $x^2-y^2 = 1$ and $x^2-y^2 = 4$.

(6) $f(x,y) = x$, and R is the region in the first quadrant bounded by curves $4x^2-y^2 = 4$, $4x^2-y^2 = 16$, $y = x$ and the x-axis.

(7) $f(x,y) = \exp(-x^2-4y^2)$, and $R = \big\{(x,y)\,\big|\,x^2+4y^2 \leqslant 1\big\}$.

(8) $f(x,y) = \exp\big(\frac{2y-x}{2x+y}\big)$, and R is the trapezoid with vertices $(0,2)$, $(1,0)$, $(4,0)$ and $(0,8)$.

Problem 3. Evaluate the triple integral $\iiint_D f(x,y,z)\,dV$ with the following f and D.

(1) $f(x,y,z) = x-y+z^2$, and D is the solid region bounded above by $z = 1+x^2+y^2$, bounded below by $z = 0$, and inside $x^2+y^2 = 4$.

(2) $f(x,y,z) = 1$, and D is the solid region bounded by $z = x^2+y^2$, $x^2+y^2 = 4$ and $z = 0$.

(3) $f(x,y,z) = 1$, and $D = \Big\{(x,y,z) \in \mathbb{R}^3\,\Big|\,\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1\Big\}$, where $a,b,c > 0$.

Problem 4. Evaluate the integral $\displaystyle\int_0^2 \big[\arctan(\pi x) - \arctan x\big]\, dx$ by converting the integral into a double integral and evaluating the double integral by changing the order of integration.

Problem 5. Evaluate the integral $\displaystyle\int_0^1 \frac{x^b - x^a}{\ln x}\, dx$, where $0 < a < b$ are constants, by converting the integral into a double integral and evaluating the double integral by changing the order of integration.

Problem 6. Evaluate the integral $\displaystyle\int_0^\infty \frac{e^{-x} - e^{-3x}}{x}\, dx$ by converting the integral into a double integral and evaluating the double integral by changing the order of integration.

Problem 7. Let a, b be positive constants. Evaluate the integral

$$\int_0^a \Big(\int_0^b \exp\big(\max\{b^2x^2, a^2y^2\}\big)\, dy\Big)\, dx\,.$$

Problem 8. Show that if $\lambda > \frac{1}{2}$, there does not exist a real-valued continuous function u such that for all x in the closed interval $[0, 1]$,

$$u(x) = 1 + \lambda \int_x^1 u(y)u(y - x)\, dy\,.$$

Hint: Assume the contrary that there exists such a function u. Integrate the equation above on the interval $[0, 1]$.

Problem 9. Let Σ denote the surface $z = 1 + xy$ and C denote the solid cylinder $x^2 + y^2 \leqslant 1$.

1. Fine the volume of the portion of C above the xy-plane and below Σ.
2. Find the surface area for the portion of Σ that is inside C.

Problem 10. A circular cylinder of radius a is circumscribed about a sphere with radius a so that the cylinder is tangent to the sphere along the equator. Two planes, each perpendicular to the axis of the cylinder, intersect the sphere and the cylinder in circles. Show that the area of that part of the sphere between the two planes is equal to the area of the part of the cylinder between the two planes.

Problem 11. Rewrite the following iterated integrals as an equivalent iterated integral in the five other orders.

(1) $\displaystyle\int_0^1 \Big[\int_y^1 \Big(\int_0^y f(x,y,z)\, dz\Big) dx\Big] dy$

(2) $\displaystyle\int_0^1 \Big[\int_y^1 \Big(\int_0^z f(x,y,z)\, dx\Big) dz\Big] dy$

(3) $\displaystyle\int_0^1 \Big[\int_0^{1-x^2} \Big(\int_0^{1-x} f(x,y,z)\, dy\Big) dz\Big] dx$

(4) $\displaystyle\int_0^3 \Big[\int_0^x \Big(\int_0^{9-x^2} f(x,y,z)\, dz\Big) dy\Big] dx$

(5) $\displaystyle\int_0^1 \Big[\int_{\sqrt{x}}^1 \Big(\int_0^{1-y} f(x,y,z)\, dz\Big) dy\Big] dx$

(6) $\displaystyle\int_{-1}^1 \Big[\int_{x^2}^1 \Big(\int_0^{1-y} f(x,y,z)\, dz\Big) dy\Big] dx$

(7) $\displaystyle\int_0^1 \Big[\int_{x^2}^{\sqrt{x}} \Big(\int_{x^2}^y f(x,y,z)\, dz\Big) dy\Big] dx$

Problem 12. Evaluate the following iterated integrals.

(1) $\int_0^{\frac{\pi}{2}}\Big[\int_0^{\frac{y}{2}}\Big(\int_0^{\frac{1}{y}}\sin y\,dz\Big)dx\Big]dy$ (2) $\int_{-5}^{5}\Big[\int_0^{\sqrt{25-x^2}}\Big(\int_0^{\frac{1}{x^2+y^2}}\sqrt{x^2+y^2}\,dz\Big)dy\Big]dx$

(3) $\int_0^{4}\Big[\int_0^{1}\Big(\int_{2y}^{2}\frac{2\cos(x^2)}{\sqrt{z}}\,dx\Big)dy\Big]dz$ (4) $\int_0^{1}\Big[\int_0^{1}\Big(\int_{x^2}^{1}xz\exp(zy^2)\,dy\Big)dx\Big]dz$

(5) $\int_0^{1}\Big[\int_{\sqrt[3]{z}}^{1}\Big(\int_0^{\ln 3}\frac{\pi e^{2x}\sin(\pi y^2)}{y^2}\,dx\Big)dy\Big]dz$ (6) $\int_0^{2}\Big[\int_0^{4-x^2}\Big(\int_0^{x}\frac{\sin(2z)}{4-z}\,dy\Big)dz\Big]dx$

Problem 13. Evaluate the iterated integral

$$\int_3^5\int_0^{1/5}\int_0^{\sqrt{25-x^2}}\sqrt{x^2+y^2}\,dydzdx+\int_{1/5}^{1/3}\int_3^{1/z}\int_0^{\sqrt{1-x^2z^2}/z}\sqrt{x^2+y^2}\,dydxdz$$

by completing the following.

1. Show that the sum of the integrals above is equal to

$$\int_3^5\int_0^{\sqrt{25-x^2}}\int_0^{1/\sqrt{x^2+y^2}}\sqrt{x^2+y^2}\,dzdydx\,.$$

2. Evaluate the integral above.

Problem 14. Evaluate the iterated integral

$$\int_1^2\int_0^{1/2}\int_0^{\sqrt{4-x^2}}\sqrt{x^2+y^2}\,dydzdx+\int_{1/2}^{1}\int_1^{1/z}\int_0^{\sqrt{1-x^2z^2}/z}\sqrt{x^2+y^2}\,dydxdz$$

by completing the following.

1. Show that the sum of the integrals above is equal to

$$\int_1^2\int_0^{\sqrt{4-x^2}}\int_0^{1/\sqrt{x^2+y^2}}\sqrt{x^2+y^2}\,dzdydx\,.$$

2. Evaluate the integral above.

Problem 15. Evaluate the sum of the following iterated integrals

$$\int_1^{e^4}\int_0^{\sqrt{4-\ln z}}\int_{\ln z}^{4-x^2}\frac{xe^y}{4-y}\,dydxdz$$

by completing the following.

1. Show that the integral above is equal to $\displaystyle\int_0^2\int_0^{4-x^2}\int_0^{e^y}\frac{xe^y}{4-y}\,dzdydx\,.$
2. Evaluate the integral above.

Problem 16. Find volume of the solid that lies under $z=x^2+y^2$ and above the region R in the xy-plane bounded by the line $y=2x$ and parabola $y=x^2$.

Problem 17. Evaluate the triple integral $\iiint_D dV$, where D is bounded by $z = x^2 + y^2$, $x^2 + y^2 = 4$ and $z = 0$.

Problem 18. Evaluate the double integral $\iint_R \arctan \frac{y}{x}\, dA$ using the polar coordinate, where

$$R = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, 1 \leqslant x^2 + y^2 \leqslant 4, 0 \leqslant y \leqslant x\right\}.$$

Problem 19. (15pts) Let R be the region in the first quadrant enclosed the straight line $y = x$, the ellipse $x^2 + 4y^2 = 4$ and the x-axis. Evaluate the double integral $\iint\limits_R \frac{y}{x}\, dA$ using the change of variables $u = x^2 + 4y^2$ and $v = y/x$.

Problem 20. Evaluate the triple integral $\iiint_D x \exp(x^2 + y^2 + z^2)\, dV$, where D is the portion of the unit ball $x^2 + y^2 + z^2 \leqslant 1$ that lies in the first octant.

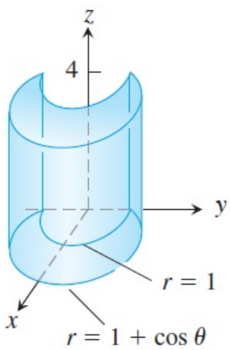
Problem 21. Evaluate the triple integral $\iiint_D \sqrt{x^2 + y^2 + z^2}\, dV$, where D is the region lying above the cone $z = \sqrt{x^2 + y^2}$ and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

Problem 22. Use the cylinder coordinate to find the volume of the ball $x^2 + y^2 + z^2 = a^2$.

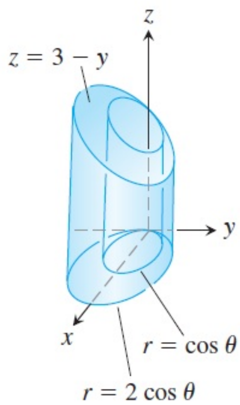
Problem 23. Use the spherical coordinate to find the volume of the cylindricality $x^2 + y^2 = r^2$, where $0 \leqslant z \leqslant h$.

Problem 24. Compute the volume of D given below using triple integrals in cylindrical coordinates.

(1) D is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.



(2) D is the solid right cylinder whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$.



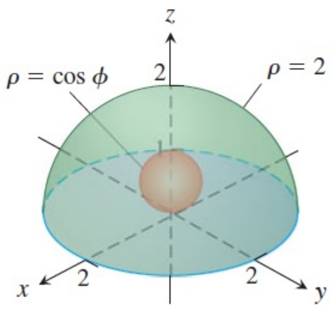
Problem 25. Evaluate the following iterated integral

$$\int_0^1 \Big[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \Big(\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} dz\Big) dy\Big] dx$$

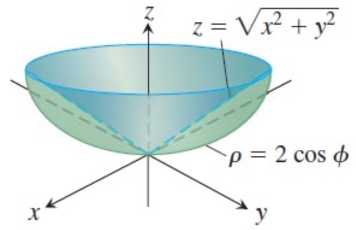
by first converting to an integral in spherical coordinates.

Problem 26. Compute the volume of D given below using triple integrals in spherical coordinates.

(1) D is the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2$, $z \geqslant 0$.



(2) D is the solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2+y^2}$.



Problem 27. Convert the integral

$$\int_{-1}^1 \Big[\int_0^{\sqrt{1-y^2}} \Big(\int_0^x (x^2+y^2)\, dz\Big) dx\Big] dy$$

to an equivalent integral in cylindrical coordinates and evaluate the result.

Problem 28. Find the integrals given below with specific change of variables.

(1) Find $\displaystyle\int_0^2 \Big(\int_{\frac{y}{2}}^{\frac{y+4}{2}} y^3(2x-y)e^{(2x-y)^2}\, dx\Big) dy$ using change of variables $x = u + \dfrac{1}{2}v$, $y = v$.

(2) Find $\displaystyle\iint_{[0,1]\times[0,1]} \frac{1}{(1+xy)\ln(xy)}\, dA$ by making the change of variables $u = xy$ and $v = y$.

(3) Find $\displaystyle\int_1^2 \Big(\int_{\frac{1}{y}}^{y} (x^2+y^2)\, dx\Big) dy + \int_2^4 \Big(\int_{\frac{y}{4}}^{\frac{4}{y}} (x^2+y^2)\, dx\Big) dy$ using change of variables $x = \dfrac{u}{v}$, $y = uv$.

(4) Find $\displaystyle\int_0^1 \Big(\int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dy\Big) dx$ using change of variables $x = u^2 - v^2$, $y = 2uv$.

(5) Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Find $\displaystyle\iint_R \Big(\sqrt{\frac{y}{x}} + \sqrt{xy}\Big)\, dA$ using the change of variables $x = \dfrac{u}{v}$, $y = uv$.

(6) Let D be the solid region in xyz-space defined by

$$D = \{(x, y, z) \mid 1 \leqslant x \leqslant 2, 0 \leqslant xy \leqslant 2, 0 \leqslant z \leqslant 1\}.$$

Find $\iiint_D (x^2y + 3xyz)\, dV$ using change of variables $u = x$, $v = xy$, $w = 3z$.

Problem 29. Evaluate the double integral $\iint_R (x+y)e^{x^2-y^2}\, dA$, where R is rectangle enclosed by the lines $x - y = 0$, $x - y = 2$, $x + y = 0$, and $x + y = 3$.

Problem 30. Let f be continuous on $[0, 1]$ and let R be the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Show that

$$\iint_R f(x+y)\, dA = \int_0^1 uf(u)\, du.$$

Problem 31. Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.
Hint: Try to make change of variables so that the computation of the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$ looks the same as the former one.

Problem 32. The spherical coordinates centered at (a, b, c) is the change of variables

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = g(\rho, \theta, \phi) = \begin{bmatrix} a + \rho \cos\theta \sin\phi \\ b + \rho \sin\theta \sin\phi \\ c + \rho \cos\phi \end{bmatrix}, \qquad (\rho, \theta, \phi) \in [0, \infty) \times [0, 2\pi] \times [0, \pi].$$

Let B denote the unit ball in the space centered at the origin.

1. Express B in terms of the spherical coordinates centered at $(0, 0, 2)$. In other words, find $g^{-1}(B)$, the corresponding region of B, in the cubic $[0, \infty) \times [0, 2\pi] \times [0, \pi]$.
2. Express the triple integral $\iiint_B \frac{1}{x^2 + y^2 + (z-2)^2}\, dV$ in terms of an iterated integral in the spherical coordinates centered at $(0, 0, 2)$.
3. Evaluate the triple integral $\iiint_B \frac{1}{x^2 + y^2 + (z-2)^2}\, dV$.
4. Similar to part 1, express B in terms of the spherical coordinates centered at $(0, 0, 1/2)$.
5. Express the triple integral $\iiint_B \frac{1}{x^2 + y^2 + (z-\frac{1}{2})^2}\, dV$ in terms of an iterated integral in the spherical coordinates centered at $(0, 0, 1/2)$, and evaluate the triple integral

$$\iiint_B \frac{1}{x^2 + y^2 + (z-\frac{1}{2})^2}\, dV.$$