# Exercise Problem Sets 8

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**Problem 1.** For positive integer $n$, let $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$. Show that $\{P_n\}_{n=0}^{\infty}$ is an orthogonal set on $[-1,1]$.

**Problem 2.** Find the cosine and sine series of the following functions.

1. $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1/2\,, \\ 0 & \text{if } 1/2 \leqslant x < 1\,. \end{cases}$
2. $f(x) = \begin{cases} 0 & \text{if } 0 < x < \pi\,, \\ x-\pi & \text{if } \pi \leqslant x < 2\pi\,. \end{cases}$
3. $f(x) = \cos x$, $0 < x < \pi/2$.
4. $f(x) = \sin x$, $0 < x < \pi$.

**Problem 3.** Find the Fourier series of the following given functions and find the value of the Fourier series at the discontinuity to conclude some identities.

1. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0\,, \\ 1 & \text{if } 0 \leqslant x < \pi\,. \end{cases}$
2. $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0\,, \\ x & \text{if } 0 \leqslant x < \pi\,. \end{cases}$.
3. $f(x) = \begin{cases} 0 & \text{if } -\pi/2 < x < 0\,, \\ \cos x & \text{if } 0 \leqslant x < \pi/2\,. \end{cases}$
4. $f(x) = \begin{cases} 1 & \text{if } -\pi/2 < x < 0\,, \\ \sin x & \text{if } 0 \leqslant x < \pi/2\,. \end{cases}$
5. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0\,, \\ e^x & \text{if } 0 \leqslant x < \pi\,. \end{cases}$
6. $f(x) = e^x$, $0 < x < \pi$.

**Problem 4.** Find the eigenvalue and eigenfunctions of the following boundary-value problem

1. $x^2y'' + xy' + \lambda y = 0$, $y(1) = y(5) = 0$.
2. $y'' + y' + \lambda y = 0$, $y(0) = y(2) = 0$.
3. $\frac{d}{dx}\big[(1+x^2)y'\big] + \frac{\lambda}{1+x^2}y = 0$, $y(0) = y(1) = 0$.

**Hint for 3**: Let $x = \tan\theta$ and then use the chain rule to rewrite the equation.

**Problem 5.** Consider the special case of the regular Sturm-Liouville problem on the interval $[a,b]$:

$$\frac{d}{dx}\big[r(x)y'\big] + \lambda p(x)y = 0\,, \qquad y'(a) = y'(b) = 0\,.$$

Prove or disprove that $\lambda = 0$ is an eigenvalue of the problem.

**Problem 6.** Use the method of separation of variables to find, if possible, product solutions for the given partial differential equation.

1. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$.
2. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.
3. $y\frac{\partial^2 u}{\partial x \partial y} + u = 0$.
4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u$.
5. $\frac{\partial^2 u}{\partial x^2} - u = \frac{\partial u}{\partial t}$.
6. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}$.