Calculus 微積分

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Chapter 0

Preliminary

0.1 Functions and Their Graphs

Definition 0.1: Real-Valued Functions of a Real Variable

Let $X, Y \subseteq \mathbb{R}$ be subsets of real numbers. A real-valued function f of a real variable x from X to Y is a correspondence that assigns to each element x in X exactly one number y in Y. Here X is called the domain of f and is usually denoted by $\mathrm{Dom}(f)$, Y is called "the" co-domain of f, the number y is called the image of x under f and is usually denoted by $f(x)$, which is called the value of f at x. The range of f, denoted by $\mathrm{Ran}(f)$, is a subset of Y consisting of all images of numbers in X. In other words,

$$\mathrm{Ran}(f) \equiv \text{the range of } f \equiv \left\{ f(x) \,\middle|\, x \in X \right\}.$$

Remark 0.2. Given a way of assignment $x \mapsto f(x)$ without specifying where x is chosen from, we still treat f as a function and $\mathrm{Dom}(f)$ is considered as the collection of all $x \in \mathbb{R}$ such that $f(x)$ is well-defined. For example, $f(x) = x + 1$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$ are both considered as functions with

$$\mathrm{Dom}(f) = \mathbb{R} \qquad \text{and} \qquad \mathrm{Dom}(g) = \mathbb{R} \backslash \{1\}.$$

Since $\mathrm{Dom}(f) \neq \mathrm{Dom}(g)$, f and g are considered as different functions even though $f(x) = g(x)$ for all $x \neq 1$.

Terminologies:

1. Explicit form of a function: $y = f(x)$;

2. Implicit form of a function: $F(x, y) = 0$. (參考影片)

Definition 0.3

A function f is a polynomial function if f takes the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \,,$$

where $a_0, a_1, a_2, \cdots, a_n$ are real numbers, called coefficients of the polynomial, and n is a non-negative integer. If $a_n \neq 0$, then a_n is called the leading coefficient, and n is called the degree of the polynomial. A rational function is the quotient of two polynomials.

Definition 0.4

The graph of the function $y = f(x)$ consists of all points $\big(x, f(x)\big)$, where x is in the domain of f. In other words,

$$\mathrm{G}(f) \equiv \text{the graph of } f \equiv \Big\{ \big(x, f(x)\big) \,\Big|\, x \in \mathrm{Dom}(f) \Big\} \,.$$

Definition 0.5: Composite Functions

Let f and g be functions. The function $f \circ g$, read f circle g, is the function defined by $(f \circ g)(x) = f\big(g(x)\big)$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words,

$$\mathrm{Dom}(f \circ g) = \big\{ x \in \mathrm{Dom}(g) \,\big|\, g(x) \in \mathrm{Dom}(f) \big\} \,.$$

0.2 Trigonometric Functions

Definition 0.6

An angle consists of an initial ray, a terminal ray and a vertex where two rays intersects. An angle is in standard position when its initial ray coincides with the positive x-axis and its vertex is at the origin. Positive angles are measured counterclockwise, and negative angles are measured clockwise.

Let θ be a central angle of a circle of radius 1. The radian measure of θ is defined to be the length of the arc of the sector.

Remark 0.7. Using radian measure of θ, the length s of a circular arc of radius r is given by $s = r\theta$.

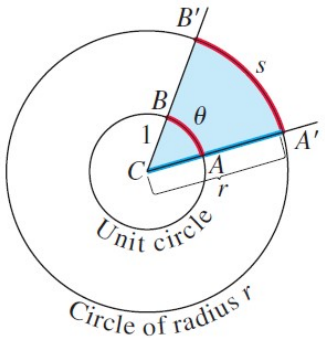


Figure 1: The radian measure of the central angle $A'CB'$ is the number $u = s/r$. For a unit circle of radius $r = 1$, u is the length of arc AB that central angle ACB cuts from the unit circle.

Remark 0.8. For a point P on the plane with Cartesian coordinate (x, y), let $r = \sqrt{x^2 + y^2}$ and θ be the angle in standard position with $\overrightarrow{OP}$ as the terminal ray. The ordered pair (r, θ) is called the ***polar coordinate*** of the point P.

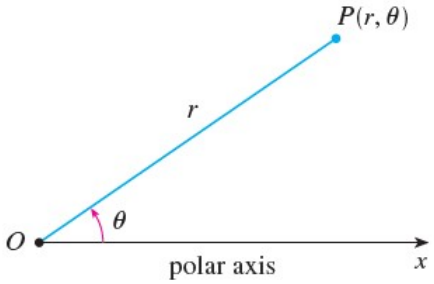


Figure 2: Polar coordinate

Definition 0.9

Let θ be an angle in standard position, and the terminal ray intersects the circle centered at the origin of radius r at point (x, y). The trigonometric functions sine, cosine, tangent, cotangent, secant and cosecant, abbreviated as sin, cos, tan, cot, sec and csc, respectively, of angle θ are defined by

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}, \quad \cot\theta = \frac{x}{y}, \quad \sec\theta = \frac{r}{x} \quad \text{and} \quad \csc\theta = \frac{r}{y},$$

provided that the quotients make sense.

Remark 0.10. Suppose that a point P has polar coordinate (r, θ). Then the Cartesian coordinate of P is $(r\cos\theta, r\sin\theta)$.

Proposition 0.11: Properties of Trigonometric Functions

1. For all real numbers θ,

$$\sin^2\theta + \cos^2\theta = 1\,, \quad 1 + \tan^2\theta = \sec^2\theta\,, \quad 1 + \cot^2\theta = \csc^2\theta\,.$$

2. For all real numbers θ,

$$\begin{aligned} &\sin(-\theta) = -\sin\theta\,, \quad \cos(-\theta) = \cos\theta\,, \quad \tan(-\theta) = -\tan\theta\,, \\ &\cot(-\theta) = -\cot\theta\,, \quad \sec(-\theta) = \sec\theta\,, \quad \csc(-\theta) = -\csc\theta\,. \end{aligned}$$

3. For all real numbers θ,

$$\begin{aligned} &\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta\,, \quad \cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta\,, \quad \tan\left(\theta + \frac{\pi}{2}\right) = -\cot\theta\,, \\ &\sin(\theta + \pi) = -\sin\theta\,, \quad \cos(\theta + \pi) = -\cos\theta\,, \quad \tan(\theta + \pi) = \tan\theta\,. \end{aligned}$$

4. (**Law of Cosines**): Let a, b, c be the length of sides of a triangle, and θ be the angle opposite to the side with length c. Then $c^2 = a^2 + b^2 - 2ab\cos\theta$.

5. (**Sum and Difference Formulas**): Let θ, ϕ be real numbers. Then

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta\,, \quad \cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi\,.$$

6. (**Double-Angle Formulas**): For all real numbers θ,

$$\sin(2\theta) = 2\sin\theta\cos\theta\,, \quad \cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta\,.$$

7. (**Half-Angle Formulas**): For all real numbers θ,

$$\cos^2\frac{\theta}{2} = \frac{1 + \cos\theta}{2}\,, \quad \sin^2\frac{\theta}{2} = \frac{1 - \cos\theta}{2}\,, \quad \tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta}\,.$$

8. (**Triple-Angle Formulas**): For all real numbers θ,

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta\,, \quad \sin(3\theta) = 3\sin\theta - 4\sin^3\theta\,.$$

9. (**Sum-to-Product Formulas**): For all real numbers θ and ϕ,

$$\begin{aligned} &\sin\theta + \sin\phi = 2\sin\frac{\theta + \phi}{2}\cos\frac{\theta - \phi}{2}\,, \quad \sin\theta - \sin\phi = 2\sin\frac{\theta - \phi}{2}\cos\frac{\theta + \phi}{2}\,, \\ &\cos\theta + \cos\phi = 2\cos\frac{\theta + \phi}{2}\cos\frac{\theta - \phi}{2}\,, \quad \cos\theta - \cos\phi = 2\sin\frac{\theta + \phi}{2}\sin\frac{\phi - \theta}{2}\,. \end{aligned}$$

Theorem 0.12: de Moivre (棣美弗)

For each real number θ and natural number n,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)\,. \tag{0.2.1}$$

Proof. Clearly (0.2.1) holds for $n = 1$. Suppose that (0.2.1) holds for $n = k$ for some natural number k. Then by the sum and difference formulas,

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)\\ &= \big[\cos(k\theta) + i\sin(k\theta)\big] \cdot (\cos\theta + i\sin\theta)\\ &= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + i\big[\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\big]\\ &= \cos[(k+1)\theta] + i\sin[(k+1)\theta]\end{aligned}$$

which shows that (0.2.1) holds for $n = k + 1$. By induction, we find that (0.2.1) holds for all natural number n. □

Theorem 0.13

Let θ be a real number and $0 \leqslant \theta < \dfrac{\pi}{2}$. Then

$$\sin\theta \leqslant \theta \leqslant \tan\theta\,. \tag{0.2.2}$$

Proof. Inequality (0.2.2) follows from the following figure

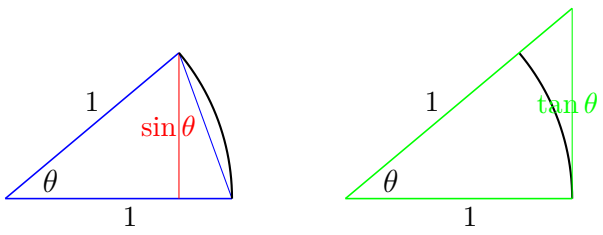


Figure 3: The area of the sector is larger than the area of the blue triangle but is smaller than the green triangle

which shows $\dfrac{1}{2}\sin\theta \leqslant \dfrac{1}{2}\theta \leqslant \dfrac{1}{2}\tan\theta$. □