*Interfaces and Free Boundaries* **16** (2014), 297–338
DOI 10.4171/IFB/321

# Global existence and decay for solutions of the Hele-Shaw flow with injection

C.H. ARTHUR CHENG
*Department of Mathematics, National Central University, Jhongli City, Taoyuan County, 32001, Taiwan ROC*
*E-mail:* cchsiao@math.ncu.edu.tw

DANIEL COUTAND
*CANPDE, Maxwell Institute for Mathematical Sciences and department of Mathematics, Heriot-Watt University, Edinburgh, EH14 4AS, UK*
*E-mail:* d.coutand@ma.hw.ac.uk

STEVE SHKOLLER
*Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, UK*
*E-mail:* shkoller@maths.ox.ac.uk

[Received 30 August 2012 and in revised form 10 March 2014]

We examine the stability and decay of the free boundary perturbations in a Hele-Shaw cell under the injection of fluid. In particular, we study the perturbations of spherical boundaries as time $t \to +\infty$. In the presence of positive surface tension, we examine both *slow* and *fast* injection rates. When fluid is injected slowly, the perturbations decay back to an expanding sphere exponentially fast, while for fast injection, the perturbation decays to an expanding sphere with an algebraic rate. In the absence of surface tension, we study the case of a constant injection rate, and prove that perturbations of the sphere decay like $(1+t)^{-1/2+\epsilon}$ for $\epsilon > 0$ small.

*2010 Mathematics Subject Classification:* Primary 35R35, 35J25, 76D27.

*Keywords:* Hele-Shaw, interface problems, stability, Hele-Shaw free boundary problems, surface tension

## 1. Introduction

### 1.1 *The problem statement*

We establish decay estimates for solutions of the Hele-Shaw equations both with and without surface tension on the free-boundary, and with injection of fluid into the cell.

1.1.1 *The case with surface tension.* With the time-dependent fluid domain denoted by $\Omega(t)$, an open subset of $\mathbb{R}^2$ with boundary $\Gamma(t)$, and for time $t \in [0, T]$, the two-dimensional Hele-Shaw equations are given by

$$\Delta p = -\mu\delta \qquad \text{in} \quad \Omega(t)\,, \tag{1a}$$
$$p = \mathrm{H} \qquad \text{on} \quad \Gamma(t)\,, \tag{1b}$$
$$\mathcal{V}(\Gamma(t)) = -\frac{\partial p}{\partial \mathrm{n}} \qquad \text{on} \quad \Gamma(t)\,, \tag{1c}$$

$$\Omega(0) = \Omega\,, \tag{1d}$$

where $\delta$ is the Dirac delta function at the origin, $p(x,t)$ denotes the fluid pressure, $\mu = \mu(t)$ denotes the rate of injection of fluid if $\mu \geqslant 0$ (or suction if $\mu \leqslant 0$), and H is the (mean) curvature of the evolving free-boundary $\Gamma(t)$. We use $\mathcal{V}(\Gamma(t))$ to denote the normal velocity of the moving free-boundary $\Gamma(t)$, and we let $n$ denote the outward-pointing unit normal on $\Gamma(t)$.

When the area of the initial domain has is $\pi$ and the injection rate $\mu \not\equiv 0$, the volume of the fluid domain $|\Omega(t)|$ can be computed as

$$|\Omega(t)| = \pi + \int_0^t \mu(s)ds =: \pi\rho(t)^2, \tag{2}$$

where $\rho(t) \equiv \sqrt{1 + \int_0^t \frac{\mu(s)}{\pi} ds} > 0$ is the radius of a ball centered at the origin. Moreover, if the initial domain $\Omega = B_1 := B(0,1)$, then the solution to (1) is given explicitly by $\Omega(t) = B(0,\rho(t))$, with pressure function

$$p(x,t) = \frac{1}{\rho(t)} - \frac{\mu(t)}{2\pi}\log\frac{|x|}{\rho(t)}.$$

We will show that under certain growth conditions on the injection rate $\mu(t)$, if
1. the initial domain $\Omega$ is sufficiently close to the unit ball $B_1$ in $\mathbb{R}^2$ with $|\Omega| = |B_1| = \pi$;
2. the center-of-mass of $\Omega$ is sufficiently close to the origin (which is the point of injection),

then $\Omega(t)$ converges to $B_\rho := B(0,\rho(t))$ as $t \to \infty$. The precise statement of our result is given below in Theorem 1.1.

Of fundamental importance to our analysis is the conversion of the second-order Poisson equation (1a) to a coupled system of first-order equations. Introducing the velocity vector $u = -\nabla p$, equation (1) can be rewritten as

$$u + \nabla p = 0 \quad \text{in} \quad \Omega(t)\,, \tag{3a}$$
$$\operatorname{div} u = \mu\delta \quad \text{in} \quad \Omega(t)\,, \tag{3b}$$
$$p = \mathrm{H} \quad \text{on} \quad \Gamma(t)\,, \tag{3c}$$
$$\mathcal{V}\big(\Gamma(t)\big) = u\cdot n \quad \text{on} \quad \Gamma(t)\,, \tag{3d}$$
$$\Omega(0) = \Omega\,. \tag{3e}$$

1.1.2 *The case with zero surface tension.* We consider the annular region $\mathbb{A} = \{x \in \mathbb{R}^2 \,:\, 1 < r < r_0\}$ where $r = |x|$ and $r_0 > 1$. Let $\Omega(t)$ be an open subset of $\mathbb{R}^2$ enclosed by an inner fixed boundary $\mathbb{S}^1$ and an outer moving free-boundary $\Gamma(t)$. For $0 \leqslant t \leqslant T$, the evolution of a two-dimensional Hele-Shaw cell is given by

$$\Delta p = 0 \quad \text{in} \quad \Omega(t)\,, \tag{4a}$$
$$p = 0 \quad \text{on} \quad \Gamma(t)\,, \tag{4b}$$
$$p = 1 \quad \text{on} \quad \mathbb{S}^1 \times [0,T)\,, \tag{4c}$$
$$\mathcal{V}\big(\Gamma(t)\big) = -\frac{\partial p}{\partial \mathrm{n}} \quad \text{on} \quad \Gamma(t)\,, \tag{4d}$$
$$\Omega(0) = \Omega\,, \tag{4e}$$

where $\mathcal{V}(\Gamma(t))$ again is used to denote the normal velocity of the moving free-boundary $\Gamma(t)$, and we let $n$ denote the outward-pointing unit normal on $\Gamma(t)$. Following the idea for the case with surface tension, we let $u = -\nabla p$ and rewrite equation (4) as

$$u + \nabla p = 0 \qquad \text{in} \quad \Omega(t)\,, \tag{5a}$$

$$\operatorname{div} u = 0 \qquad \text{in} \quad \Omega(t)\,, \tag{5b}$$

$$p = 0 \qquad \text{on} \quad \Gamma(t)\,, \tag{5c}$$

$$p = 1 \qquad \text{on} \quad \mathbb{S}^1 \times (0,T)\,, \tag{5d}$$

$$\mathcal{V}\big(\Gamma(t)\big) = u \cdot n \qquad \text{on} \quad \Gamma(t)\,, \tag{5e}$$

$$\Omega(0) = \Omega\,. \tag{5f}$$

Suppose that we choose our annular region as the initial domain, so that $\Omega = \mathbb{A}$; in this case, the solution to the Hele-Shaw equation (4) is radially symmetric, and the domain $\Omega(t)$ occupied by the fluid continues as an annular region, bounded by $r = 1$ and $r = \rho(t)$ for some $\rho(t) > 1$, which we specify later in (8). We prove that if we choose an initial domain $\Omega$ to be a small perturbation of the annular region $\mathbb{A}$ in $\mathbb{R}^2$, with the constraint that $|\Omega| = |\mathbb{A}| = \pi(r_0^2 - 1)$, then the distance between the moving boundary $\Gamma(t)$ and the surface $B_\rho \equiv B(0,\rho)$ decays to zero.

The equation that the radius $\rho(t)$ must satisfy is derived as follows. Recall that $\rho(t)$ is the radius of the expanding ball of the unperturbed solution $(\bar{p}, \Omega(t))$ corresponding to initial data $\Omega(t)|_{t=0} = \mathbb{A}$. This solution consists of a radially symmetric pressure function $\bar{p}$ and an annular region $\Omega(t) = B_\rho \backslash B_1$. Since $\rho'$ is the rate-of-change of the radius, $\rho(t)$ must satisfy

$$\rho' = -\frac{\partial \bar{p}}{\partial n} \qquad \text{on} \quad \partial B_\rho$$

which states that the rate-of-change of the radius is the normal velocity of the moving surface $\Gamma(t)$. Nevertheless, since $\bar{p}$ is harmonic with $\bar{p} = 1$ on $\mathbb{S}^1$ and $\bar{p} = 0$ on $\partial B_\rho$, we must have that

$$\bar{p}(x,t) \equiv 1 - \frac{\log|x|}{\log \rho(t)} \qquad \text{in} \quad B_\rho \backslash B_1 .$$

As a consequence,

$$-\frac{\partial \bar{p}}{\partial n}(t)\Big|_{r=\rho(t)} = \frac{1}{\rho(t)\log \rho(t)} \qquad \text{on} \quad \partial B_\rho \tag{6}$$

which implies that

$$\rho(t) \log \rho(t) \rho'(t) = 1. \tag{7}$$

Since $\displaystyle\int x \log x \, dx = \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + C$,

$$\frac{1}{2}\rho^2 \log \rho - \frac{1}{4}\rho^2 = t + \frac{1}{2} r_0^2 \log r_0 - \frac{1}{4} r_0^2 ;$$

thus letting $\mathrm{C}_0 \equiv r_0^2 (2\log r_0 - 1)$, we find that $\rho$ satisfies

$$\rho^2 (2 \log \rho - 1) = 4t + \mathrm{C}_0 . \tag{8}$$

While we cannot explicitly solve (8) for $\rho$, it suffices to find a lower and upper bound (in order to understand the decay rate). First, for all $\alpha > 0$,

$$C_\alpha \rho^{2+\alpha} \geqslant 2\rho^2 \log \rho \geqslant 4t + \mathrm{C}_0$$

for some constant $C_\alpha$; thus for some $\lambda_{\alpha,r_0} > 0$,

$$\rho(t) \geqslant \lambda_{\alpha,r_0}(1+t)^{1/(2+\alpha)} \qquad \forall\, t > 0. \tag{9}$$

On the other hand,

$$\rho^2 \log \frac{r_0^2}{e} \leqslant \rho^2 \log \frac{\rho^2}{e} = \rho^2(2\log\rho - 1) = 4t + \mathrm{C}_0;$$

thus for some $\Lambda_0 > 0$,

$$\rho(t) \leqslant \Lambda_0(1+t)^{1/2} \qquad \forall\, t > 0. \tag{10}$$

### 1.2 *Some prior results*

1.2.1 *Surface tension on the free boundary.* In the case that fluid is *not* being injected into the Hele-Shaw cell (that is, $\mu = 0$), Constantin & Pugh [5] established the stability and exponential decay of solutions of (1) using the methods of complex analysis. Friedman & Reitich [14] also establish this stability result. In [1], Chen studied a two-phase Hele-Shaw problem with surface tension, and established well-posedness using the energy method coupled with certain pointwise estimates from the theory of harmonic functions; moreover, he proved that solutions exist for all time if the initial interface is a sufficiently small perturbation of equilibrium. Classical short-time solutions to related problems have been obtained by Escher and Simonett [11] in multiple space dimensions. In any dimension greater than or equal to two, Escher and Simonett [13] established global existence and stability near spherical shapes using center manifold theory.

In the case of a constant fluid injection-rate $\mu > 0$, Prokert [17] and Vondenhoff [20] established global existence results.

1.2.2 *The case of zero surface tension.* For the case of fluid injection with zero surface tension, weak solutions were considered by Elliott & Janokowski [9] (see also Elliott & Ockendon [10]) and Gustafsson [2], and Escher & Simonett [12] proved the existence of unique and smooth solutions to the Hele-Shaw equations on a finite-time interval. For the case of fluid suction when $p < 0$, cusp formation occurs [16]. When $p > 0$, global-in-time weak solutions and their regularity have been studied in [9], [15] using variational inequalities and in [2], [3], and [4] for Lipschitz initial boundaries using comparison function ideas. For convex initial boundaries, global existence of regular solutions was established in [8].

### 1.3 *Statement of our main results*

We propose a simple methodology (equally applicable in three space dimensions) for establishing global existence and *decay* to equilibrium for Hele-Shaw cells with positive fluid injection rates (time-dependent in the case of positive surface tension on the free boundary). We use an Arbitrary

Eulerian Lagrangian (ALE) formulation or *harmonic coordinates* to transform the free-boundary problem (3) to a system of PDE on a fixed domain. This ALE transformation depends on the signed height function $\mathbf{h}$ measuring the signed distance between the moving surface $\Gamma(t)$ and the expanding sphere $\partial B_{\rho(t)}$. To be more precise, $\mathbf{h} : \mathbb{S}^1 \times (0, T) \to \mathbb{R}$ is given by

$$\mathbf{h}(\xi, t) = \sup \big\{ r > 0 \,\big|\, r\xi \in \Omega(t) \big\} - \rho(t) \,.$$

Moreover, when $\Gamma(t)$ remains a graph over $\partial B_\rho$, the sign distance function $\mathbf{h}$ can be written as

$$\mathbf{h}\Big(\frac{\mathbf{x}}{|\mathbf{x}|}, t\Big) = \begin{cases} -\mathrm{dist}(\mathbf{x}, \partial B_{\rho(t)}) & \forall\, \mathbf{x} \in \Gamma(t) \cap B_{\rho(t)}, \\ \ \ \mathrm{dist}(\mathbf{x}, \partial B_{\rho(t)}) & \forall\, \mathbf{x} \in \Gamma(t) \backslash B_{\rho(t)}. \end{cases} \tag{11}$$

The idea behind the proof is the construction of a (total) norm, denoted by $|||\cdot|||_T$, which consists of a norm $\|\cdot\|_X$ associated to energy estimates in sufficiently high-regularity Sobolev spaces, and a norm $\|\cdot\|_Y$ associated to decay estimates in weaker topologies. The total norm is given by

$$|||\cdot|||_T = \sup_{t\in[0,T]} \big( \|\cdot\|_X + \mathfrak{D}(t)\|\cdot\|_Y \big)$$

for some function $\mathfrak{D}(t) \to \infty$ as $t \to \infty$. We shall prove that $\mathbf{h}$ satisfies

$$|||\mathbf{h}|||_T \leqslant C\epsilon + |||\mathbf{h}|||_T^{2p} \tag{12}$$

for some integer $p$. The inequality (12) implies that if $\epsilon$ (an upper bound for a norm of the initial data $\mathbf{h}$) is sufficiently small, then $|||\mathbf{h}|||_T$ stays small for all $t \in [0, T]$; a standard continuation argument shows provides global existence and the fact $\mathfrak{D}(t)$ in front of the lower-order norm $\|\cdot\|_Y$ gives the decay to equilibrium.

1.3.1 *Notation.* For each $t \geqslant 0$, we define the moment $\mathfrak{M}_{\Omega(t)}$ of $\Omega(t)$ by

$$\mathfrak{M}_{\Omega(t)} = \int_{\Omega(t)} (x, y)\, dA \,,$$

where $dA$ denotes the infinitesimal area element, and $(x, y)$ is the vector coordinate of a point in $\Omega(t)$. In particular, $\mathfrak{M}_{\Omega(t)}$ is a vector containing the $x$-moment and $y$-moment.

We define the center-of-mass $\mathfrak{C}_{\Omega(t)}$ by

$$\mathfrak{C}_{\Omega(t)} = \frac{\mathfrak{M}_{\Omega(t)}}{|\Omega(t)|} \,,$$

where $|\Omega(t)|$ denotes the area of $\Omega(t)$.

1.3.2 *The case of positive surface tension on the free boundary.*

THEOREM 1.1 (Decay for slow injection) Let $(p, \Omega(t))$ be the solution to (1), $\mathbf{h}$ denote the signed distance between $\Gamma(t) = \partial\Omega(t)$ and $\partial B_\rho$ with $\rho(t)$ defined by (2), and the center-of-mass $\mathfrak{C}_{\Omega(t)} = (0, 0)$. If the injection rate $\mu \geqslant 0$ is such that the corresponding radius $\rho(t)$ satisfies

$$\sup_{t>0} \frac{\rho^{(k)}(t)(1+t)^k}{\rho(t)} < \infty \ \ k = 1, 2, \quad \text{and} \quad \sup_{t>0} \frac{\rho(t)}{(1+t)^\alpha} < \infty \text{ for some } \alpha \leqslant \frac{1}{3} \,, \tag{13}$$

then there exists an $\epsilon > 0$ sufficiently small, such that the solution to (1) exists for all $t \geqslant 0$ provided that $\|\mathbf{h}_0\|_{H^6(\mathbb{S}^1)} \leqslant \epsilon$. Moreover, the signed distance $\mathbf{h}$ decays to zero, and

$$\|\mathbf{h}(t)\|_{H^{2.5}(\mathbb{S}^1)} \leqslant C\rho(t)^{-2}e^{-\beta d(t)} \qquad \forall\, t > 0 \tag{14}$$

for some constant $C > 0$ and $\beta \in (0, 7/8)$, where $d(t) = \displaystyle\int_0^t \frac{6ds}{\rho(s)^3}$.

THEOREM 1.2 (Decay for fast injection) Let $(p, \Omega(t))$ denote the solution to (1), and let $\mathbf{h}$ denote the signed distance between $\Gamma(t) = \partial\Omega(t)$ and $\partial B_\rho$ with $\rho(t)$ defined by (2). Suppose that $\mu \geqslant 0$ is such that the corresponding $\rho(t)$ satisfies

$$\sup_{t>0} \frac{\rho'(t)(1+t)}{\rho(t)} < \infty \quad \text{and} \quad \nu \equiv \sup\left\{\alpha \,\middle|\, \sup_{t>0} \frac{(1+t)^\alpha}{\rho(t)} < \infty\right\} > \frac{3}{8} \tag{15}$$

as well as one of the following conditions:
(1) $\rho'' \leqslant 0$ or
(2) $\log \rho'$ has small enough total variation.
Then there exists an $\epsilon > 0$ sufficiently small, such that the solution to (1) exists for all $t \geqslant 0$ provided that $\|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)} \leqslant \epsilon$ with

$$\mathrm{K} = \max\left\{6, \left[\frac{64\nu - 21}{16\nu - 6}\right] + 1\right\}.$$

Moreover, the signed distance $\mathbf{h}$ decays to zero, and

$$\|\mathbf{h}(t)\|_{H^{2.5}(\mathbb{S}^1)} \leqslant C\frac{\sqrt{1+t}}{\rho(t)^2} \qquad \forall\, t > 0 \tag{16}$$

for some constant $C > 0$.

In the following discussion, we define

$$\mathfrak{D}(t) = \begin{cases} \rho(t)^2 e^{\beta d(t)} & \text{if (13) is satisfied,} \\ \dfrac{\rho(t)^2}{\sqrt{1+t}} & \text{if (15) is satisfied.} \end{cases} \tag{17}$$

Then (14) and (16) can be summarized as

$$\mathfrak{D}(t)\|\mathbf{h}(t)\|_{H^{2.5}(\mathbb{S}^1)} \leqslant C \qquad \forall\, t > 0 \tag{18}$$

for some constant $C > 0$.

REMARK 1.3 In Theorem 1.2, the smallness condition $\|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)} \leqslant \epsilon$ guarantees that the center-of-mass $\mathcal{C}_{\Omega(0)}$ cannot be too far from the point of injection $(0, 0)$.

REMARK 1.4 Assumption (13) limits the injection rate of the fluid. In this case, $\mathcal{C}_{\Omega(0)} = (0, 0)$ is crucial for the stability result. On the other hand, when (15) is valid, the injection is fast enough so that even $\mathcal{C}_{\Omega(0)}$ is close to the origin, then the stability result holds.

REMARK 1.5 Suppose that (15) is satisfied. By Gronwall's inequality, the condition

$$\rho'(t) \leqslant C\rho(t)(1+t)^{-1} \quad \text{for some constant } C > 0,$$

implies that $\nu < \infty$. Then for each $0 < \varepsilon \ll 1$, there exists $C_1$ and $C_2$ such that

$$\frac{(1+t)^{\nu-\varepsilon}}{\rho(t)} \leqslant \frac{1}{C_1} \quad \text{and} \quad \frac{(1+t)^{\nu+\varepsilon}}{\rho(t)} \geqslant \frac{1}{C_2} \qquad \forall\, t > 0$$

so that

$$C_1(1+t)^{\nu-\varepsilon} \leqslant \rho(t) \leqslant C_2(1+t)^{\nu+\varepsilon} \qquad \forall\, t > 0. \tag{19}$$

Consequently, $\rho(t)^{2-\frac{1}{2\nu^-}} \leqslant C\,\mathfrak{D}(t)$, where we use the notation $\nu^{\pm}$ to denote $\nu \pm \varepsilon$ whenever $0 < \varepsilon \ll 1$. We will make crucial use of the integrability of certain functions of these bounds.

When $\mu = 0$, $\rho(t) = 1$ so (13) holds. Theorem 1.1 implies that the distance from the moving boundary $\Gamma(t)$ to the boundary of the equilibrium state (wherein the boundary is the unit circle) decays to zero exponentially, which is result that was obtained by [1], [5], [14], and [13], but with more conditions on the data.

When $\mu$ is a positive constant, we have $\rho(t) = \sqrt{1+\frac{\mu}{\pi}t}$. For this case, Vondenhoff [20] provides a decay estimate for a rescaled function:

$$\Big\|\frac{\mathbf{h}}{\rho}\Big\|_{H^6(\mathbb{S}^1)} \leqslant C\rho^{-\alpha} \qquad \forall\, \alpha \in (0,1) \text{ and } t > 0\,,$$

but this, in fact, shows that $\|\mathbf{h}\|_{H^6(\mathbb{S}^1)} \sim t^{(1-\alpha)/2}$ for large $t > 0$ and hence the perturbation may actually grow in time.

In contrast, for the case of constant injection, our Theorem 1.2 shows that

$$\|\mathbf{h}(t)\|_{H^{2.5}(\mathbb{S}^1)} \leqslant C\rho(t)^{-1} \qquad \forall\, t > 0$$

which is the first result proving the decay rate of the actual height function of the perturbation of the sphere. With the rescaled variable,

$$\Big\|\frac{\mathbf{h}}{\rho}\Big\|_{H^{2.5}(\mathbb{S}^1)} \leqslant C\rho(t)^{-2} \qquad \forall\, t > 0\,.$$

Moreover, our theorem applies to time-dependent injection rates, and thus generalizes the results of [17] and [20].

### 1.3.3 *The case of zero surface tension on the free boundary.*

THEOREM 1.6 Let $(p, \Omega(t))$ denote the solution to (4), let $\mathbf{h} : \mathbb{S}^1 \times (0,T) \to \mathbb{R}$ defined by (11) denote the signed distance between $\Gamma(t)$ and $\partial B_{\rho(t)}$, with $\rho(t)$ given as the solution to (8). Then there exists an $\epsilon > 0$ sufficiently small, such that the solution to (4) exists for all $t \geqslant 0$ provided that $\|\mathbf{h}_0\|_{H^3(\mathbb{S}^1)} \leqslant \epsilon$. Moreover, the signed distance $\mathbf{h}$ decays to zero, and

$$\|\mathbf{h}(t)\|_{H^3(\mathbb{S}^1)} \leqslant C\rho(t)^{-1} \qquad \forall\, t > 0 \tag{20}$$

for some constant $C > 0$.

### 1.4 *Outline*

We start our discussion for the case with surface tension. In Section 2, we derive the evolution equation for the signed height function $\mathbf{h}$ and define the corresponding ALE map $\psi$ (or harmonic coordinate), which we use to pull-back the equations onto the fixed domain $B_1$. The total norm $|||\cdot|||_T$ and the basic bootstrapping assumption that we use to establish (12) are introduced in Section 3. In Section 4, we derive some inequalities, fundamental for our subsequent analysis.

In Section 5, an estimate of the type

$$\mathfrak{D}(t)\|\mathbf{h}(t)\|_{H^{2.5}(\mathbb{S}^1)} \leqslant C\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||^2_T + |||\mathbf{h}|||^8_T\Big] \qquad \forall\, t \in [0,T] \tag{21}$$

is derived using the decay property of the linearized problem, where $[0,T]$ is the time interval of the existence of the solution. We combine this with energy estimates for $\mathbf{h}$ in a higher-order Sobolev space in Section 6, from which we obtain

$$\Big[\int_0^T \frac{1}{\rho(t)^3}\|\mathbf{h}(t)\|^2_{H^{\mathrm{K}+1.5}(\mathbb{S}^1)}dt\Big]^{\frac{1}{2}} + \sup_{t\in[0,T]}\Big[\|\mathbf{h}(t)\|_{H^{\mathrm{K}}(\mathbb{S}^1)} + \rho(t)\sqrt{\rho'(t)}\|\mathbf{h}(t)\|_{H^{\mathrm{K}-1}(\mathbb{S}^1)}\Big]$$
$$\leqslant C_\delta M(\|\mathbf{h}_0\|_{H^6(\mathbb{S}^1)}) + C_\delta |||\mathbf{h}|||^2_T \mathcal{P}\big(|||\mathbf{h}|||^2_T\big) + \delta|||\mathbf{h}|||_T \tag{22}$$

for some polynomial $\mathcal{P}$, and then show how the decay estimate (21) together with our energy estimate (22) leads to Theorem 1.1 and 1.2. Section 7 contributes to the proof of Theorem 1.6, the case with zero surface tension.

## 2. Fixing the domain for the case of positive surface tension

### 2.1 *The signed height function* h

We let $\mathbb{S}^1 = \partial B_1$ denote the boundary of the unit ball, and parametrize $\mathbb{S}^1$ using the usual angular variable $\theta$. For each $\theta \in \mathbb{S}^1$, let $\mathbf{h}(\theta,t)$ denote the signed distance from $\Gamma(t)$ to $\partial B(0,\rho(t))$, where the sign of $\mathbf{h}$ is taken positive if for $\mathbf{x}(\theta,t) \in \Gamma(t)$, $|\mathbf{x}(\theta,t)| > \rho(t)$ and taken negative if $|\mathbf{x}(\theta,t)| < \rho(t)$. In other words, $\Gamma(t)$ can be parametrized by the equation

$$\mathbf{x}(\theta,t) = (x(\theta,t), y(\theta,t)) = \big(\rho(t) + \mathbf{h}(\theta,t)\big)\mathrm{N}(\theta) \qquad \forall\,\theta \in \mathbb{S}^1\,, \tag{23}$$

where $\mathrm{N}(\theta) = (\cos\theta, \sin\theta)$ is the outward-pointing unit normal to $B_1$.

Let $\mathrm{T}(\theta) = (-\sin\theta, \cos\theta)$ be the tangent vector on $\mathbb{S}^1$. Then the outward-pointing unit normal at the point $(x(\theta,t), y(\theta,t))$ is

$$n(\theta,t) = \mathbf{J}_{\mathbf{h}}^{-1}(\theta,t)\big[\big(\rho(t) + \mathbf{h}(\theta,t)\big)\mathrm{N}(\theta) - \mathbf{h}_\theta(\theta,t)\mathrm{T}(\theta)\big], \tag{24}$$

where $\mathbf{J}_{\mathbf{h}}(\theta,t) = \sqrt{(\rho(t) + \mathbf{h}(\theta,t))^2 + \mathbf{h}_\theta(\theta,t)^2}$.

### 2.2 *The mean curvature in terms of* h

In addition, we note that with respect to the height function $\mathbf{h}$, the mean curvature is given by

$$\begin{aligned}\mathrm{H}_{\mathbf{h}}(\mathbf{x}) &= \frac{-(\rho+\mathbf{h})\mathbf{h}_{\theta\theta} + (\rho+\mathbf{h})^2 + 2\mathbf{h}_\theta^2}{\big[(\rho+\mathbf{h})^2 + \mathbf{h}_\theta^2\big]^{3/2}} \\ &= \frac{-(\rho+\mathbf{h})\mathbf{h}_{\theta\theta} + \mathbf{J}_{\mathbf{h}}^2 + \mathbf{h}_\theta^2}{\mathbf{J}_{\mathbf{h}}^3} \qquad \text{on} \quad \mathbb{S}^1\times(0,T).\end{aligned} \tag{25}$$

### 2.3 *A divergence-free velocity*

When $\mathbf{h}_0 = 0$ so that $\Omega = B_1$, the solution to the Hele-Shaw equation (3) is given by $\Omega(t) = B(0, \rho(t))$,

$$\bar{p}(x,t) = \frac{1}{\rho(t)} - \frac{\mu(t)}{2\pi} \log \frac{|x|}{\rho(t)}, \quad \bar{u}(x,t) = \frac{\mu(t)}{2\pi} \frac{x}{|x|^2}.$$

In order to have a divergence-free velocity field, we introduce the new variables $\mathbf{u} = u - \bar{u}$ and $\mathbf{p} = p - \bar{p}$, so that (3) is converted to

$$\mathbf{u} + \nabla \mathbf{p} = 0 \qquad \text{in} \quad \Omega(t), \tag{26a}$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in} \quad \Omega(t), \tag{26b}$$

$$\mathbf{p} = \mathrm{H} - \bar{p} \qquad \text{on} \quad \Gamma(t). \tag{26c}$$

### 2.4 *The ALE formulation (harmonic coordinates)*

Let $\psi(\cdot,t)$ denote the ALE mapping, taking $B_1$ to $\Omega(t)$, defined as the solution to the elliptic equation

$$\Delta \psi = 0 \qquad \text{in} \quad B_1, \tag{27a}$$

$$\psi = (\rho + \mathbf{h})\mathrm{N} \qquad \text{on} \quad \mathbb{S}^1, \tag{27b}$$

where the Laplacian has to be read component-wise on $\psi$. For sufficiently small perturbations $\mathbf{h}$, elliptic estimates and the inverse function theorem show that $\psi(t) = \psi(\cdot,t)$ is a diffeomorphism, and $\Omega(t) = \psi(t)(B_1)$. By introducing the ALE variables $\mathbf{v} = \mathbf{u} \circ \psi$, $\mathbf{q} = \mathbf{p} \circ \psi$, $\bar{\mathbf{q}} = \bar{p} \circ \psi$, and $\mathrm{A} = \nabla \psi^{-1}$, we find that (26) can be rewritten on the fixed domain as

$$\mathbf{v}^i + \mathrm{A}_i^j \mathbf{q},_j = 0 \qquad \text{in} \quad B_1, \tag{28a}$$

$$\mathrm{A}_i^j \mathbf{v}^i,_j = 0 \qquad \text{in} \quad B_1, \tag{28b}$$

$$\mathbf{q} = \mathrm{H}_{\mathbf{h}}(\mathbf{x}) - \bar{\mathbf{q}} \qquad \text{on} \quad \mathbb{S}^1, \tag{28c}$$

where $\mathrm{H}_{\mathbf{h}}(\mathbf{x})$ is given by (25). We are employing the Einstein summation convention, wherein repeated indices are summed from 1 to 2, and we write $F,_k$ to mean $\dfrac{\partial F}{\partial x_k}$.

### 2.5 *The evolution equation of* $\mathbf{h}$

We determine the evolution equation for $\mathbf{h}$ in order to close the system. The boundary condition (27b) implies that

$$\psi_t \cdot n = (\rho' + \mathbf{h}_t)(\mathrm{N} \cdot n) \qquad \text{on} \quad \mathbb{S}^1. \tag{29}$$

Since $\psi_t \cdot n$ is the normal velocity of the free boundary $\Gamma(t) = \psi(t)(\mathbb{S}^1)$, we must have

$$\psi_t \cdot n = (u \circ \mathbf{x}) \cdot n \qquad \text{on} \quad \mathbb{S}^1.$$

Therefore, by defining

$$\mathfrak{n}(\theta,t) := \mathrm{N}(\theta) - \frac{\mathbf{h}_\theta(\theta,t)}{\rho(t) + \mathbf{h}(\theta,t)} \mathrm{T}(\theta) = \frac{\mathbf{J}_{\mathbf{h}}(\theta,t)}{\rho(t) + \mathbf{h}(\theta,t)} n(\theta,t), \tag{30}$$

we obtain that

$$\mathbf{h}_t + \rho' = \frac{1}{\mathrm{N}\cdot n}\Big[\mathbf{v}\cdot n + \big((\bar{u}\circ\mathbf{x})\cdot n\big)\Big] = \mathbf{v}\cdot\mathfrak{n} + \frac{\mu}{2\pi(\rho+\mathbf{h})}$$

or equivalently,

$$\mathbf{h}_t = \mathbf{v}\cdot\mathfrak{n} - \frac{\rho'\mathbf{h}}{\rho+\mathbf{h}} \qquad \text{on} \quad \mathbb{S}^1\times(0,T)\,. \tag{31}$$

### 2.6 *The vector* $\mathrm{JA}^{\mathrm{T}}\mathrm{N}$

Let $\mathrm{J} = \det(\nabla\psi)$. Since $\psi(\theta,t) = \big(\rho(t)+\mathbf{h}(\theta,t)\big)\mathrm{N}(\theta)$ on $\mathbb{S}^1$, we find that

$$\begin{aligned}\mathrm{JA}^{\mathrm{T}}\mathrm{N} &= \begin{bmatrix}\psi^2{}_{,2} & -\psi^2{}_{,1}\\ -\psi^1{}_{,2} & \psi^1{}_{,1}\end{bmatrix}\begin{bmatrix}\mathrm{N}_1\\ \mathrm{N}_2\end{bmatrix} = \begin{bmatrix}\psi^2{}_{,1} & \psi^2{}_{,2}\\ -\psi^1{}_{,1} & -\psi^1{}_{,2}\end{bmatrix}\begin{bmatrix}\mathrm{T}_1\\ \mathrm{T}_2\end{bmatrix} = \Big[\frac{\partial\psi^2}{\partial\mathrm{T}}, -\frac{\partial\psi^1}{\partial\mathrm{T}}\Big]^{\mathrm{T}}\\ &= \Big[\frac{\partial\psi^2}{\partial\theta}, -\frac{\partial\psi^1}{\partial\theta}\Big]^{\mathrm{T}} = \big[\mathbf{h}_\theta\sin\theta + (\rho+\mathbf{h})\cos\theta, -\mathbf{h}_\theta\cos\theta + (\rho+\mathbf{h})\sin\theta\big]^{\mathrm{T}}\\ &= (\rho+\mathbf{h})\mathrm{N} - \mathbf{h}_\theta\mathrm{T} = (\rho+\mathbf{h})\mathfrak{n}(\theta).\end{aligned} \tag{32}$$

### 2.7 *Linearization about the unperturbed state* $\mathbf{h}\equiv 0$

When $\mathbf{h}_0 = 0$, $\mathbf{h} = 0$ for all $t > 0$, in which case, $\psi(x,t) = \rho(t)x$ and $\mathrm{A} = \rho^{-1}\mathrm{Id}$. Therefore, we may decompose (28a,b) into a linear term and a nonlinear remainder as

$$\mathbf{v} + \rho^{-1}\nabla\mathbf{q} = f_1 \qquad \text{in} \quad B_1\times(0,T), \tag{33a}$$

$$\mathrm{div}\mathbf{v} = f_2 \qquad \text{in} \quad B_1\times(0,T), \tag{33b}$$

where

$$f_1^i = (\rho^{-1}\delta_i^j - \mathrm{A}_i^j)\mathbf{q}_{,j} \quad \text{and} \quad f_2 = (\delta_i^j - \rho\mathrm{A}_i^j)\mathbf{v}^i{}_{,j}\,.$$

In order to determine the linear operator associated to the boundary condition (28c), we multiply both sides of (28c) by $\rho^{-2}(\rho+\mathbf{h})^{-1}\mathbf{J}_{\mathbf{h}}^3$, and find that

$$\mathbf{q} = \frac{1}{\rho^2}\Big[-\mathbf{h}_{\theta\theta} + \rho + \mathbf{h} + \frac{\mathbf{h}_\theta^2}{\rho+\mathbf{h}}\Big] - (\gamma+1)\bar{\mathbf{q}} - \gamma\mathbf{q}, \tag{34}$$

where

$$\begin{aligned}\gamma &= \rho^{-2}(\rho+\mathbf{h})^{-1}\mathbf{J}_{\mathbf{h}}^3 - 1 = \frac{\mathbf{J}_{\mathbf{h}}^6 - \rho^4(\rho+\mathbf{h})^2}{\rho^2(\rho+\mathbf{h})\big[\mathbf{J}_{\mathbf{h}}^3 + \rho^2(\rho+\mathbf{h})\big]}\\ &= \frac{(\rho+\mathbf{h})^2(4\rho^3\mathbf{h} + 6\rho^2\mathbf{h}^2 + 4\rho\mathbf{h}^3 + \mathbf{h}^4) + 3(\rho+\mathbf{h})^4\mathbf{h}_\theta^2 + 3(\rho+\mathbf{h})^2\mathbf{h}_\theta^4 + \mathbf{h}_\theta^6}{\rho^2(\rho+\mathbf{h})\big[\mathbf{J}_{\mathbf{h}}^3 + \rho^2(\rho+\mathbf{h})\big]}.\end{aligned}$$

We remark that

$$\gamma = \frac{2\mathbf{h}}{\rho} + \mathrm{g},$$

for some function g of $\mathbf{h}$ and $\mathbf{h}_\theta$ satisfying

$$\|g\|_{H^k(\mathbb{S}^1)} \leqslant C\rho^{-2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1}(\mathbb{S}^1)}.$$

Since

$$\bar{\mathbf{q}} = \frac{1}{\rho} - \frac{\mu}{2\pi}\log\Big(\frac{\rho+\mathbf{h}}{\rho}\Big) = \frac{1}{\rho} - \frac{\mu}{2\pi}\log\Big(1+\frac{\mathbf{h}}{\rho}\Big),$$

a Taylor expansion in $\mathbf{h}/\rho \ll 1$ shows that $\bar{\mathbf{q}} \approx \frac{1}{\rho} - \frac{\mu\mathbf{h}}{2\pi\rho} = \rho^{-1} - \rho'\mathbf{h}$; thus by writing $\bar{\mathbf{q}} = \bar{\bar{\mathbf{q}}} + \rho^{-1} - \rho'\mathbf{h}$, where

$$\bar{\bar{\mathbf{q}}} = -\frac{\mu}{2\pi}\log\Big(1+\frac{\mathbf{h}}{\rho}\Big) + \frac{\mu\mathbf{h}}{2\pi\rho} = -\frac{\mu}{2\pi}\Big[\log\Big(1+\frac{\mathbf{h}}{\rho}\Big) - \frac{\mathbf{h}}{\rho}\Big],$$

we may write (34) as

$$\mathbf{q} = -\frac{1}{\rho^2}\big[\mathbf{h}_{\theta\theta}+\mathbf{h}\big] + \rho'\mathbf{h} - \gamma\mathbf{q} + \mathcal{E} \qquad \text{on} \quad \mathbb{S}^1\times(0,T), \tag{33c}$$

where $\mathcal{E}$ denotes an error-term given by

$$\mathcal{E} = \frac{\mathbf{h}_\theta^2}{\rho^2(\rho+\mathbf{h})} - (\gamma+1)\bar{\bar{\mathbf{q}}} + \gamma\rho'\mathbf{h} - \frac{g}{\rho}.$$

### 2.8 *A homogeneous version of* (33)

We can now define a new velocity field which is divergence-free. Let $f_1 = w + \rho^{-1}\nabla r$, where r is the zero-average solution to the following elliptic equation

$$\rho^{-1}\Delta r = \operatorname{div} f_1 - f_2 \qquad \text{in} \quad B_1\,, \tag{35a}$$

$$\rho^{-1}\frac{\partial r}{\partial N} = f_1\cdot N - \frac{1}{2\pi}\int_{B_1} f_2\,dx \qquad \text{on} \quad \mathbb{S}^1\,. \tag{35b}$$

We remark that the solvability of (35) is guaranteed by the solvability condition

$$\int_{B_1}(\operatorname{div} f_1 - f_2)\,dx = \int_0^{2\pi}\Big(f_1\cdot N - \frac{1}{2\pi}\int_{B_1} f_2\,dx\Big)d\theta.$$

Let $v = \mathbf{v} - w$, and $q = \mathbf{q} - r$. By (35a) we find that $\operatorname{div} w_1 = f_2$; thus (33) implies that

$$v + \rho^{-1}\nabla q = 0 \qquad \text{in} \quad B_1\times(0,T)\,, \tag{36a}$$

$$\operatorname{div} v = 0 \qquad \text{in} \quad B_1\times(0,T)\,, \tag{36b}$$

$$q = -\frac{\mathbf{h}_{\theta\theta}}{\rho^2} - \frac{\mathbf{h}}{\rho^2} + \rho'\mathbf{h} - \gamma\mathbf{q} + G \qquad \text{on} \quad \mathbb{S}^1\times(0,T)\,, \tag{36c}$$

where $G = \mathcal{E} - r$. We note that on $\mathbb{S}^1$,

$$\begin{aligned} v\cdot\mathfrak{N} = (\mathbf{v}-w)\cdot\mathfrak{N} &= \mathbf{h}_t + \frac{\rho'\mathbf{h}}{\rho+\mathbf{h}} + \frac{\mathbf{h}_\theta(w\cdot T)}{\rho+\mathbf{h}} - (w\cdot N) \\ &= \mathbf{h}_t + \frac{\rho'\mathbf{h}}{\rho+\mathbf{h}} + \frac{\mathbf{h}_\theta(w\cdot T)}{\rho+\mathbf{h}} - \frac{1}{2\pi}\int_{B_1} f_2\,dx. \end{aligned} \tag{37}$$

## 3. The total norm for the case of positive surface tension

We first define the total norm $|||\cdot|||_T$, used to establish (12), as follows:

$$|||\mathbf{h}|||_T \equiv \left[\int_0^T \frac{1}{\rho(t)^3}\|\mathbf{h}(t)\|^2_{H^{\mathrm{K}+1.5}(\mathbb{S}^1)}dt\right]^{\frac{1}{2}} + \sup_{t\in[0,T]}\left[\|\mathbf{h}(t)\|_{H^{\mathrm{K}}(\mathbb{S}^1)} + \rho(t)\sqrt{\rho'(t)}\|\mathbf{h}(t)\|_{H^{K-1}(\mathbb{S}^1)} + \mathfrak{D}(t)\|\mathbf{h}(t)\|_{H^{2.5}(\mathbb{S}^1)}\right], \quad (38)$$

where $\mathrm{K}=6$ if (13) is satisfied or $\mathrm{K}=\max\left\{6,\left[\dfrac{64\nu-21}{16\nu-6}\right]+1\right\}$ if (15) is satisfied, and we recall that $\mathfrak{D}(t)$ is defined in (17) by

$$\mathfrak{D}(t) = \begin{cases} \rho(t)^2 e^{\beta d(t)} & \text{if (13) is satisfied,} \\ \dfrac{\rho(t)^2}{\sqrt{1+t}} & \text{if (15) is satisfied.} \end{cases}$$

We remark that once the boundedness of $|||\mathbf{h}|||_T$ is established, $\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}$ decays to zero at the rate $\mathfrak{D}(t)^{-1}$. Throughout the rest of the paper, we assume that $\rho$ satisfies the condition (13) or (15); that is,

$$\sup_{t>0}\frac{\rho^{(k)}(t)(1+t)^k}{\rho(t)} < \infty \ k=1,2, \quad \text{and} \quad \sup_{t>0}\frac{\rho(t)}{(1+t)^\alpha} < \infty \text{ for some } \alpha \leqslant \frac{1}{3} \quad (13)$$

or

$$\sup_{t>0}\frac{\rho'(t)(1+t)}{\rho(t)} < \infty \quad \text{and} \quad \nu \equiv \sup\left\{\alpha \,\Big|\, \sup_{t>0}\frac{(1+t)^\alpha}{\rho(t)} < \infty\right\} > \frac{3}{8}. \quad (15)$$

Moreover, we make the following basic assumption: for $t\in[0,T]$ and for sufficiently small positive constants $\epsilon$ and $\sigma$ to be made precise later,

$$\|\mathbf{h}(t)\|_{H^{\mathrm{K}}(\mathbb{S}^1)} + \mathfrak{D}(t)\|\mathbf{h}(t)\|_{H^{2.5}(\mathbb{S}^1)} \leqslant \sigma \ll 1 \quad \text{and} \quad \|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)} \leqslant \epsilon \ll 1. \quad (39)$$

(We will prove that (39) indeed holds whenever the initial data is sufficiently small.)

## 4. A priori estimates for the case of positive surface tension

### 4.1 *Estimates of* $\psi$, A *and* J

Under assumption (39), elliptic estimates show that

$$\|\nabla\psi-\rho\mathrm{Id}\|_{L^\infty(B_1)} \leqslant C\|\nabla\psi-\rho\mathrm{Id}\|_{H^{1.5}(B_1)} \leqslant C\|\mathbf{h}\|_{H^2(\mathbb{S}^1)} \quad (40)$$

and for $0\leqslant k\leqslant \mathrm{K}-1.5$,

$$\|D^2\psi\|_{H^k(B_1)} \leqslant C\|\mathbf{h}\|_{H^{k+1.5}(\mathbb{S}^1)}. \quad (41)$$

Estimate (40) implies that

$$\begin{aligned}\|\rho\mathrm{A}-\mathrm{Id}\|_{L^\infty(B_1)} &= \|(\rho\mathrm{Id}-\nabla\psi)\mathrm{A}\|_{L^\infty(B_1)} \leqslant C\|\mathrm{A}\|_{L^\infty(B_1)}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)} \\ &\leqslant C\rho^{-1}\left[\|\rho\mathrm{A}-\mathrm{Id}\|_{L^\infty(B_1)}+1\right]\|\mathbf{h}\|_{H^2(\mathbb{S}^1)};\end{aligned}$$

thus under assumption (39),

$$\|\rho \mathrm{A} - \mathrm{Id}\|_{L^\infty(B_1)} \leqslant C\rho^{-1}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}. \tag{42}$$

We note that (42) further suggests that

$$\|\mathrm{A}\|_{L^\infty(B_1)} \leqslant C\rho^{-1}. \tag{43}$$

Since $DA = -AD^2\psi A$, (41) and (43) together imply that for $0 \leqslant k \leqslant \mathrm{K} - 1.5$,

$$\|D\mathrm{A}\|_{H^k(B_1)} \leqslant C\rho^{-2}\|\mathbf{h}\|_{H^{k+1.5}(\mathbb{S}^1)}. \tag{44}$$

Furthermore, with $\mathrm{J} \equiv \det(\nabla\psi)$, inequality (40) implies that

$$\|\mathrm{J} - \rho^2\|_{L^\infty(B_1)} \leqslant C\rho\|\mathbf{h}\|_{H^2(\mathbb{S}^1)} \tag{45}$$

and (42) and (44) together with (45) show that

$$\|D\mathrm{J}\|_{H^k(B_1)} \leqslant C\rho\|\mathbf{h}\|_{H^{k+1.5}(\mathbb{S}^1)}, \tag{46}$$

from which it follows that for $0 \leqslant k \leqslant \mathrm{K} - 1.5$,

$$\|D(\mathrm{JA})\|_{H^k(B_1)} \leqslant C\|\mathbf{h}\|_{H^{k+1.5}(\mathbb{S}^1)}. \tag{47}$$

### 4.2 *Estimates of* $\bar{\mathbf{q}}$ *and* $\bar{\bar{\mathbf{q}}}$

Recall that $\bar{\bar{\mathbf{q}}} = -\frac{\mu}{2\pi}\Big[\log\Big(1+\frac{\mathbf{h}}{\rho}\Big) - \frac{\mathbf{h}}{\rho}\Big]$. Let $\bar{\partial} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{\partial}{\partial\theta}$ denote the differentiation in the tangential direction. Since $\big|\log(1+x) - x\big| \leqslant x^2$ if $|x| < 0.5$, and

$$\bar{\partial}\bar{\bar{\mathbf{q}}} = \frac{\rho'\mathbf{h}\bar{\partial}\mathbf{h}}{\rho+\mathbf{h}} \qquad \text{on} \quad \mathbb{S}^1,$$

we find that with the basic assumption (39), for $1 \leqslant k \leqslant \mathrm{K} - 2.5$,

$$\|\bar{\bar{\mathbf{q}}}\|_{H^k(\mathbb{S}^1)} \leqslant C\rho'\rho^{-1}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^k(\mathbb{S}^1)}. \tag{48}$$

Since $\bar{\mathbf{q}} = \bar{\bar{\mathbf{q}}} + \rho^{-1} - \rho'\mathbf{h}$,

$$\|\bar{\mathbf{q}}\|_{H^k(\mathbb{S}^1)} \leqslant C\big[\rho^{-1}\big(1 + \|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^k(\mathbb{S}^1)}\big) + \rho'\|\mathbf{h}\|_{H^k(\mathbb{S}^1)}\big]. \tag{49}$$

### 4.3 *Estimates of* $f_1$ *and* $f_2$

By the definition of $f_1$ and $f_2$, for $k \geqslant 1$,

$$\|f_1\|_{H^k(B_1)} \leqslant C\rho^{-2}\big[\|\mathbf{h}\|_{H^{k+1}(\mathbb{S}^1)}\|\mathbf{q}\|_{H^{1.5}(B_1)} + \|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{q}\|_{H^{k+1}(B_1)}\big] \tag{50}$$

and

$$\|f_2\|_{H^{k-1}(B_1)} \leqslant \begin{cases} C\rho^{-1}\Big[\|\mathbf{h}\|_{H^{k+1}(\mathbb{S}^1)}\|\mathbf{v}\|_{H^{1.5}(B_1)} + \|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{v}\|_{H^k(B_1)}\Big] & \text{if } k > 1, \\ C\rho^{-1}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{v}\|_{H^1(B_1)} & \text{if } k = 1. \end{cases} \tag{51}$$

### 4.4 *Estimates of* $\mathbf{q}$

Before proceeding to the estimate of $\mathbf{q}$, we remark that since

$$\mathcal{E} = \frac{\mathbf{h}_\theta^2}{\rho^2(\rho+\mathbf{h})} - (\gamma+1)\bar{\bar{\mathbf{q}}} + \gamma\rho'\mathbf{h} - \frac{1}{\rho}\mathrm{g}\,,$$

by (48) we obtain that

$$\|\mathcal{E}\|_{H^{k+0.5}(\mathbb{S}^1)} \leqslant C\Big[\frac{1}{\rho^3}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1.5}(\mathbb{S}^1)} + \frac{\rho'}{\rho}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)} + \frac{\rho'}{\rho^2}\|\mathbf{h}\|^2_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1.5}(\mathbb{S}^1)}\Big]. \quad (52)$$

Note that $\mathbf{q}$ satisfies

$$\Delta\mathbf{q} = \rho\big[\operatorname{div} f_1 - f_2\big] \quad \text{in} \quad B_1\times(0,T), \quad (53a)$$

$$\mathbf{q} = -\frac{\mathbf{h}_{\theta\theta}}{\rho^2} - \frac{\mathbf{h}}{\rho^2} + \rho'\mathbf{h} - \gamma\mathbf{q} + \mathcal{E} \quad \text{on} \quad \mathbb{S}^1\times(0,T). \quad (53b)$$

Elliptic estimates together with (49)–(52) then show that

$$\begin{aligned}\|\mathbf{q}\|_{H^2(B_1)} &\leqslant C\Big[\rho\big(\|f_1\|_{H^1(B_1)} + \|f_2\|_{L^2(B_1)}\big) + \rho^{-2}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{1.5}(\mathbb{S}^1)} \\ &\qquad + \|\gamma\mathbf{q}\|_{H^{1.5}(\mathbb{S}^1)} + \|\mathcal{E}\|_{H^{1.5}(\mathbb{S}^1)}\Big] \\ &\leqslant C\Big[\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\big(\|\mathbf{q}\|_{H^2(B_1)} + \|\mathbf{v}\|_{H^1(B_1)}\big) + \rho^{-2}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{1.5}(\mathbb{S}^1)} \\ &\qquad + \|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{1.5}(\mathbb{S}^1)} + \rho^{-2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} + \|\gamma\mathbf{q}\|_{H^{1.5}(\mathbb{S}^1)}\Big].\end{aligned}$$

Since

$$\begin{aligned}\|\mathbf{v}\|_{H^1(B_1)} &\leqslant \rho^{-1}\|\mathbf{q}\|_{H^2(B_1)} + \|f_1\|_{H^1(B_1)} \leqslant C\big(\rho^{-1} + \|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\big)\|\mathbf{q}\|_{H^2(B_1)}, \\ \|\gamma\mathbf{q}\|_{H^{1.5}(\mathbb{S}^1)} &\leqslant C\Big[\rho^{-1}\|\mathbf{h}\|_{H^{1.5}(\mathbb{S}^1)} + \rho^{-2}\|\mathbf{h}\|^2_{H^{2.5}(\mathbb{S}^1)}\Big]\|\mathbf{q}\|_{H^2(B_1)},\end{aligned}$$

by assumption (39) we find that

$$\|\mathbf{q}\|_{H^2(B_1)} \leqslant C\Big[\rho^{-2}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{1.5}(\mathbb{S}^1)}\Big].$$

Similarly, elliptic estimates also show that for integers $1 \leqslant k \leqslant \mathrm{K}-1$,

$$\begin{aligned}\|\mathbf{q}\|_{H^{k+1}(B_1)} \leqslant C\Big[&\rho^{-1}\|\mathbf{h}\|_{H^{k+1}(\mathbb{S}^1)}\big(\|\mathbf{q}\|_{H^{1.5}(B_1)} + \|\mathbf{v}\|_{H^{1.5}(B_1)}\big) \\ &+ \rho^{-1}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\big(\|\mathbf{q}\|_{H^{k+1}(B_1)} + \|\mathbf{v}\|_{H^k(B_1)}\big) \\ &+ \rho^{-2}\|\mathbf{h}\|_{H^{k+2.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)} + \|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^k(\mathbb{S}^1)} \\ &+ \rho^{-2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1.5}(\mathbb{S}^1)}\Big] + C\|\gamma\mathbf{q}\|_{H^{k+0.5}(\mathbb{S}^1)}.\end{aligned}$$

Since

$$\begin{aligned}\|\mathbf{v}\|_{H^k(B_1)} &\leqslant \rho^{-1}\|\mathbf{q}\|_{H^{k+1}(B_1)} + \|f_1\|_{H^k(B_1)}\\ &\leqslant C\rho^{-1}\Big[\|\mathbf{q}\|_{H^{k+1}(B_1)} + \|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1}(\mathbb{S}^1)}\Big],\end{aligned}$$

and

$$\begin{aligned}\|\gamma\mathbf{q}\|_{H^{k+0.5}(\mathbb{S}^1)} &\leqslant C\Big[\|\gamma\|_{H^1(\mathbb{S}^1)}\|\mathbf{q}\|_{H^{k+1}(B_1)} + \|\gamma\|_{H^{k+0.5}(\mathbb{S}^1)}\|\mathbf{q}\|_{H^{1.5}(B_1)}\Big]\\ &\leqslant C\Big[\big(\rho^{-1}\|\mathbf{h}\|_{H^1(\mathbb{S}^1)} + \rho^{-2}\|\mathbf{h}\|^2_{H^2(\mathbb{S}^1)}\big)\|\mathbf{q}\|_{H^{k+1}(B_1)}\\ &\qquad + \rho^{-1}\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)} + \rho^{-2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1.5}(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)}\Big],\end{aligned} \tag{54}$$

we conclude that

$$\begin{aligned}\|\mathbf{q}\|_{H^{k+1}(B_1)} \leqslant C\Big[&\rho^{-2}\|\mathbf{h}\|_{H^{k+1}(\mathbb{S}^1)}\|\mathbf{q}\|_{H^3(B_1)} + \rho^{-1}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1}(\mathbb{S}^1)}\\ &+ \rho^{-2}\|\mathbf{h}\|_{H^{k+2.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)} + \|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^k(\mathbb{S}^1)}\\ &+ \rho^{-2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1.5}(\mathbb{S}^1)}\Big].\end{aligned}$$

In particular, when $k = 2$, using our basic assumption (39), we obtain that

$$\begin{aligned}\|\mathbf{q}\|_{H^3(B_1)} \leqslant C\Big[&\rho^{-1}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)}\|\mathbf{h}\|_{H^3(\mathbb{S}^1)}\\ &+ \rho^{-2}\|\mathbf{h}\|_{H^{4.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} + \|\mathbf{h}\|^2_{H^2(\mathbb{S}^1)}\\ &+ \rho^{-2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)}\Big]\end{aligned}$$

and can hence be made small by (39). This, in turn, implies that

$$\begin{aligned}\|\mathbf{v}\|_{H^k(B_1)} + \rho^{-1}\|\mathbf{q}\|_{H^{k+1}(B_1)} &\leqslant C\Big[\rho^{-2}\|\mathbf{h}\|_{H^{4.5}(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1}(\mathbb{S}^1)} + \rho^{-3}\|\mathbf{h}\|_{H^{k+2.5}(\mathbb{S}^1)}\\ &\qquad + \frac{\rho'}{\rho}\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)} + \rho^{-3}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1.5}(\mathbb{S}^1)}\Big]\\ &\leqslant C\rho^{-1}\Big[\|\mathbf{h}\|_{H^{k+2.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)}\Big].\end{aligned} \tag{55}$$

Consequently,

$$\|f_1\|_{H^k(B_1)} + \|f_2\|_{H^{k-1}(B_1)} \leqslant C\rho^{-2}\big[\|\mathbf{h}\|_{H^{k+2.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)}\big] \tag{56}$$

for $1 \leqslant k \leqslant \mathrm{K}-2$, where we have used the basic assumption $\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} \leqslant \sigma$.

On the other hand, we note that (27a) implies that $(\nabla\psi)\mathbf{v} + \nabla\mathbf{q} = 0$; thus if $k = 1$, by (39) we obtain that

$$\begin{aligned}\|\nabla\mathbf{q}\|_{H^1(B_1)} &\leqslant \|\nabla\psi\mathbf{v}\|_{H^1(B_1)}\\ &\leqslant C\Big[\|\nabla\psi\|_{L^\infty(B_1)}\|\mathbf{v}\|_{H^1(B_1)} + \|D\psi\|_{W^{1,4}(B_1)}\|\mathbf{v}\|_{L^4(B_1)}\Big]\\ &\leqslant C\Big[\rho\|\mathbf{v}\|_{H^1(B_1)} + \|D^2\psi\|_{L^4(B_1)}\|\mathbf{v}\|_{H^1(B_1)}\Big] \leqslant C\rho\|\mathbf{v}\|_{H^1(B_1)}\,.\end{aligned}$$

Moreover, if $1 < k \leqslant \mathrm{K}-1$, by the Sobolev embedding $H^k(B_1) \subseteq L^\infty(B_1)$ we obtain that

$$\begin{aligned}\|\nabla \mathbf{q}\|_{H^k(B_1)} &\leqslant C\Big[\|\nabla\psi\|_{L^\infty(B_1)}\|\mathbf{v}\|_{H^k(B_1)} + \|\nabla\psi\|_{H^k(B_1)}\|\mathbf{v}\|_{L^\infty(B_1)}\Big]\\ &\leqslant C\Big[\rho\|\mathbf{v}\|_{H^k(B_1)} + \|\nabla^2\psi\|_{H^{k-1}(B_1)}\|A^{\mathrm{T}}\nabla\mathbf{q}\|_{L^\infty(B_1)}\Big]\\ &\leqslant C\Big[\rho\|\mathbf{v}\|_{H^k(B_1)} + \rho^{-1}\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)}\|\nabla\mathbf{q}\|_{H^k(B_1)}\Big]\\ &\leqslant C\Big[\rho\|\mathbf{v}\|_{H^k(B_1)} + \rho^{-1}\|\mathbf{h}\|_{H^{\mathrm{K}-0.5}(\mathbb{S}^1)}\|\nabla\mathbf{q}\|_{H^k(B_1)}\Big]\\ &\leqslant C\Big[\rho\|\mathbf{v}\|_{H^k(B_1)} + \sigma\|\nabla\mathbf{q}\|_{H^k(B_1)}\Big];\end{aligned}$$

thus since $\sigma \ll 1$, $\|\nabla\mathbf{q}\|_{H^k(B_1)} \leqslant C\rho\|\mathbf{v}\|_{H^k(B_1)}$ for $1 < k \leqslant \mathrm{K}-1$. As a consequence, for $1 \leqslant k \leqslant \mathrm{K}-1$,

$$\|\nabla\mathbf{q}\|_{H^k(B_1)} \leqslant C\rho\|\mathbf{v}\|_{H^k(B_1)}. \tag{57}$$

### 4.5 *Estimates of* r *and* w

Applying elliptic estimates to (35), using (56) we find that

$$\begin{aligned}\|\mathrm{r}\|_{H^{k+1}(B_1)} &\leqslant C\rho\Big[\|f_1\|_{H^k(B_1)} + \|f_2\|_{H^{k-1}(B_1)}\Big]\\ &\leqslant C\rho^{-1}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}\Big[\|\mathbf{h}\|_{H^{k+2.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)}\Big].\end{aligned} \tag{58}$$

Since $\mathrm{w} = f_1 - \rho^{-1}\nabla\mathrm{r}$, we also obtain that

$$\begin{aligned}\|\mathrm{w}\|_{H^k(B_1)} &\leqslant \|f_1\|_{H^k(B_1)} + \rho^{-1}\|\mathrm{r}\|_{H^{k+1}(B_1)}\\ &\leqslant C\rho^{-2}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}\Big[\|\mathbf{h}\|_{H^{k+2.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)}\Big].\end{aligned} \tag{59}$$

### 4.6 *Estimates of* $v$ *and* $q$ *in terms of* $\mathbf{h}$

Since $v = \mathbf{v} - \mathrm{w}$ and $q = \mathbf{q} - \mathrm{r}$,

$$\begin{aligned}&\|v\|_{H^k(B_1)} + \rho^{-1}\|q\|_{H^{k+1}(B_1)}\\ &\quad\leqslant \|\mathbf{v}\|_{H^k(B_1)} + \|\mathrm{w}\|_{H^k(B_1)} + \rho^{-1}\|\mathbf{q}\|_{H^{k+1}(B_1)} + \rho^{-1}\|\mathrm{r}\|_{H^{k+1}(B_1)}\\ &\quad\leqslant C\rho^{-1}\Big[\|\mathbf{h}\|_{H^{k+2.5}(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)}\Big].\end{aligned} \tag{60}$$

### 4.7 *Elliptic estimates for the height function* $\mathbf{h}$

We may rewrite the boundary condition (36c) as

$$-\mathbf{h}_{\theta\theta} + \rho^2\rho'\mathbf{h} = \mathbf{h} + \rho^2\big[\mathbf{q} + \gamma\mathbf{q} - \mathcal{E}\big].$$

Elliptic estimates then imply that for $1 \leqslant k \leqslant K-1$,

$$\|\mathbf{h}\|^2_{H^{k+2.5}(\mathbb{S}^1)} + \rho^2\rho' \|\mathbf{h}\|^2_{H^{k+1.5}(\mathbb{S}^1)} \leqslant C\|\mathbf{h}\|^2_{H^{k+0.5}(\mathbb{S}^1)} + \rho^4\Big[\|\mathbf{q}\|^2_{H^{k+1}(B_1)} + \|\gamma\mathbf{q}\|^2_{H^{k+0.6}(\mathbb{S}^1)} + \|\mathcal{E}\|^2_{H^{k+0.5}(\mathbb{S}^1)}\Big];$$

thus by (52), (54) and (57) together with the smallness of $\|\mathbf{h}\|_{H^K(\mathbb{S}^1)}$, as well as the boundedness of $\rho'/\rho$, we find that

$$\|\mathbf{h}\|^2_{H^{k+2.5}(\mathbb{S}^1)} + \rho^2\rho' \|\mathbf{h}\|^2_{H^{k+1.5}(\mathbb{S}^1)} \leqslant C\Big[\rho^6\|\mathbf{v}\|^2_{H^k(B_1)} + \|\mathbf{h}\|^2_{H^{k+0.5}(\mathbb{S}^1)} + \rho^2\rho'^2)\|\mathbf{h}\|^2_{H^2(\mathbb{S}^1)}\Big].$$

It then follows that for $1 \leqslant k \leqslant \mathrm{K}-1$,

$$\rho^{-3}\|\mathbf{h}\|^2_{H^{k+2.5}(\mathbb{S}^1)} \leqslant C\Big[\rho^3\|\mathbf{v}\|^2_{H^k(B_1)} + \big(\rho^{-3} + \rho^{-1}\rho'^2\big)\|\mathbf{h}\|^2_{H^{2.5}(\mathbb{S}^1)}\Big]. \tag{61}$$

## 4.8 *An elliptic estimate via the Hodge decomposition*

Recall that $\bar\partial = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ is the tangential differential operator. The following lemma is used to recover the full regularity of a function given that all its tangential derivatives are estimated. In the following, we let $\mathbb{A}_r$ denote the annular region enclosed by $\partial B_r$ and $\mathbb{S}^1$.

LEMMA 4.1 Let $\Omega = B_1$ or $\Omega = \mathbb{A}_{1/\rho}$ for some $\rho \geqslant 2$. Suppose that $v \in L^2(\Omega)$ satisfies $\operatorname{div} v = \operatorname{curl} v = 0$, and the tangential derivatives $\bar\partial^\ell v \in L^2(\Omega)$ for all $l = 1, 2, \ldots, k$, for any integer $k \geqslant 1$. Then $v \in H^k(\Omega)$ and satisfies

$$\|v\|_{H^k(\Omega)} \leqslant C \sum_{j=0}^{k} \|\bar\partial^j v\|_{L^2(\Omega)} \tag{62}$$

for some constant $C$ independent of $\rho$.

*Proof.* We first prove (62) for the case $\Omega = B_1$ or $\Omega = \mathbb{A}_{1/2}$. Using elliptic estimates which follow from the Hodge decomposition (see, for example, §5.9 in [18]), we conclude that

$$\begin{aligned}
\|v\|_{H^k(\Omega)} &\leqslant C\Big[\|v\|_{L^2(\Omega)} + \|\operatorname{curl} v\|_{H^{k-1}(\Omega)} + \|\operatorname{div} v\|_{H^{k-1}(\Omega)} + \|v\cdot \mathrm{N}\|_{H^{k-0.5}(\partial\Omega)}\Big] \\
&\leqslant C\Big[\|v\|_{L^2(\Omega)} + \|v\cdot \mathrm{N}\|_{H^{k-0.5}(\partial\Omega)}\Big] \leqslant C\Big[\|v\|_{L^2(\Omega)} + \|\bar\partial(v\cdot \mathrm{N})\|_{H^{k-1.5}(\partial\Omega)}\Big] \\
&\leqslant C\Big[\|v\|_{L^2(\Omega)} + \|\bar\partial v\cdot \mathrm{N}\|_{H^{k-1.5}(\partial\Omega)} + \|v\cdot \mathrm{T}\|_{H^{k-1.5}(\partial\Omega)}\Big] \\
&\leqslant C\Big[\|v\|_{L^2(\Omega)} + \|\bar\partial^2 v\cdot \mathrm{N}\|_{H^{k-2.5}(\partial\Omega)} + \|\bar\partial v\cdot \mathrm{T}\|_{H^{k-2.5}(\partial\Omega)} + \|\bar\partial(v\cdot \mathrm{T})\|_{H^{k-2.5}(\partial\Omega)}\Big] \\
&\leqslant C\Big[\|v\|_{L^2(\Omega)} + \|\bar\partial v\cdot \mathrm{N}\|_{H^{k-2.5}(\partial\Omega)} + \|\bar\partial^2 v\cdot \mathrm{N}\|_{H^{k-2.5}(\partial\Omega)} + \|\bar\partial v\cdot \mathrm{T}\|_{H^{k-2.5}(\partial\Omega)}\Big].
\end{aligned}$$

By an induction argument, we find that

$$\|v\|_{H^k(\Omega)} \leqslant C\Big[\|v\|_{L^2(\Omega)} + \sum_{j=1}^{k} \|\bar\partial^j v\cdot \mathrm{N}\|_{H^{-0.5}(\partial\Omega)} + \|\bar\partial^k v\cdot \mathrm{T}\|_{H^{-0.5}(\partial\Omega)}\Big]. \tag{63}$$

On the other hand, the fact that $\operatorname{div} v = \operatorname{curl} v = 0$ suggests that

$$\operatorname{div}(\bar\partial v) = (-y v^1_x + x v^1_y)_x + (-y v^2_x + x v^2_y)_y = -y(\operatorname{div} v)_x + x(\operatorname{div} v)_y + \operatorname{curl} v = 0$$

and

$$\operatorname{curl}(\bar\partial v) = (-y v^2_x + x v^2_y)_x - (-y v^1_x + x v^1_y)_y = x(\operatorname{curl} v)_y - y(\operatorname{curl} v)_x + \operatorname{div} v = 0\,;$$

thus $\operatorname{div}(\bar\partial^j v) = \operatorname{curl}(\bar\partial^j v) = 0$ for all $j \geqslant 0$. The classical normal trace estimate asserts that if a vector field $v \in L^2(\Omega)$ and $\operatorname{div} v \in L^2(\Omega)$, then the trace of $v \cdot \mathrm{n} \in H^{-\frac12}(\partial\Omega)$ and satisfies

$$\|v \cdot \mathrm{N}\|_{H^{-0.5}(\partial\Omega)} \leqslant C\Big[\|v\|_{L^2(\Omega)} + \|\operatorname{div} v\|_{L^2(\Omega)}\Big]\,.$$

See [19] for a proof. Similarly, we have that

$$\|v \cdot \mathrm{T}\|_{H^{-0.5}(\partial\Omega)} \leqslant C\Big[\|v\|_{L^2(\Omega)} + \|\operatorname{curl} v\|_{L^2(\Omega)}\Big]\,.$$

From these inequalities, we see that for $j \geqslant 0$,

$$\begin{aligned} &\|\bar\partial^j v \cdot \mathrm{N}\|_{H^{-0.5}(\partial\Omega)} + \|\bar\partial^j v \cdot \mathrm{T}\|_{H^{-0.5}(\partial\Omega)} \\ &\quad \leqslant C\Big[\|\bar\partial^j v\|_{L^2(\Omega)} + \|\operatorname{div}(\bar\partial^j v)\|_{L^2(\Omega)} + \|\operatorname{curl}(\bar\partial^j v)\|_{L^2(\Omega)}\Big] \leqslant C\|\bar\partial^j v\|_{L^2(\Omega)} \end{aligned}$$

which, combined with (63), gives (62).

Now suppose the annulus is given by $\Omega = \mathbb{A}_{1/\rho}$ where $\rho = \rho(t)$ for some $\rho \geqslant 2$. Consider the change of variables $(r, \theta) \to (r', \theta')$ given by

$$r' = 2(1 - \rho^{-1})(r - 1) + 1, \quad \theta' = \theta.$$

Then

$$\frac{\partial}{\partial r'} = \frac{1}{2(1-\rho^{-1})}\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \theta'} = \frac{\partial}{\partial \theta}\,.$$

If $\widetilde v(r, \theta) = v(r', \theta')$, then

$$\|v\|_{H^k(\mathbb{A}_{1/\rho})} \leqslant C_1\|\widetilde v\|_{H^k(\mathbb{A}_{1/2})} \quad \text{and} \quad \|\bar\partial^\ell \widetilde v\|_{L^2(\mathbb{A}_{1/2})} \leqslant C_2\|\bar\partial^\ell v\|_{L^2(\mathbb{A}_{1/\rho})} \tag{64}$$

for some constant $C_1$ and $C_2$ independent of $\rho$ if $\rho \geqslant 2$. As a consequence,

$$\|v\|_{H^k(\mathbb{A}_{1/\rho})} \leqslant C\|\widetilde v\|_{H^k(\mathbb{A}_{1/2})} \leqslant C\sum_{\ell=0}^{k}\|\bar\partial^\ell \widetilde v\|_{L^2(\mathbb{A}_{1/2})} \leqslant C\sum_{\ell=0}^{k}\|\bar\partial^\ell v\|_{L^2(\mathbb{A}_{1/\rho})}$$

which concludes the lemma. □

## 5. Decay estimates for $\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}$ for positive surface tension

### 5.1 *The Fourier representation of solutions*

Letting $\mathrm{G}_1 = \mathrm{G} + \gamma \mathbf{q}$, we express $\mathbf{h}$ and $\mathrm{G}_1$ in terms of their Fourier series:

$$\mathbf{h}(\theta, t) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \widehat{\mathbf{h}}_k(t) e^{ik\theta}, \quad \mathrm{G}_1(\theta, t) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \widehat{\mathrm{G}_1}_k(t) e^{ik\theta}.$$

Since $q$ is harmonic in $B_1$ with the Dirichlet boundary condition (36c), the Poisson integral formula shows that

$$q(r, \theta, t) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \Big[ \frac{|k|^2 - 1}{\rho^2} \widehat{\mathbf{h}}_k + \rho' \widehat{\mathbf{h}}_k + \widehat{\mathrm{G}_1}_k \Big] r^{|k|} e^{ik\theta} \quad \text{for} \quad r < 1\,.$$

Taking the inner product of (36) with the vector $\mathfrak{N}$, by (37) and the fact that $\dfrac{\partial q}{\partial \mathrm{N}} = \dfrac{\partial q}{\partial r}\Big|_{r=1}$, we find that

$$\mathbf{h}_t + \frac{\rho' \mathbf{h}}{\rho + \mathbf{h}} + \frac{1}{\rho} \frac{\partial q}{\partial r}\Big|_{r=1} = \frac{\mathbf{h}_\theta \nabla q \cdot \mathrm{T}}{\rho(\rho + \mathbf{h})}\Big|_{r=1} - \frac{\mathbf{h}_\theta (\mathrm{w} \cdot \mathrm{T})}{\rho + \mathbf{h}}\Big|_{r=1} + \frac{1}{2\pi} \int_{B_1} f_2 \, dx$$

or

$$\begin{aligned} \mathbf{h}_t + \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \Big[ \frac{|k|(|k|^2 - 1)}{\rho^3} + (|k| + 1) \frac{\rho'}{\rho} \Big] \widehat{\mathbf{h}}_k e^{ik\theta} &= -\frac{1}{\sqrt{2\pi}\,\rho} \sum_{k\neq 0} |k| \widehat{\mathrm{G}_1}_k e^{ik\theta} \\ &\quad + \frac{\mathbf{h}_\theta}{\rho(\rho + \mathbf{h})} \frac{\partial q}{\partial \theta}\Big|_{r=1} - \frac{\mathbf{h}_\theta (\mathrm{w} \cdot \mathrm{T})}{\rho + \mathbf{h}}\Big|_{r=1} + \frac{1}{2\pi} \int_{B_1} f_2 \, dx - \frac{\rho' \mathbf{h}^2}{\rho(\rho + \mathbf{h})}. \end{aligned} \tag{65}$$

Therefore, for $k \neq 0, \pm 1$ we have

$$\frac{d\widehat{\mathbf{h}}_k}{dt} + \Big[ \frac{|k|(|k|^2 - 1)}{\rho^3} + (|k| + 1) \frac{\rho'}{\rho} \Big] \widehat{\mathbf{h}}_k = \widehat{R}_k, \tag{66}$$

where $\widehat{R}_k$ is the Fourier coefficient of $R$ which is defined as the right-hand side of (65). Note that estimates (51) and (58) then suggest that

$$\begin{aligned} \|R\|_{H^{2.5}(\mathbb{S}^1)} &\leqslant C\rho^{-1} \Big[ \|\mathrm{G} + \gamma \mathbf{q}\|_{H^{3.5}(\mathbb{S}^1)} + \|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} \|q\|_{H^4(B_1)} \\ &\qquad + \|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)} \|q\|_{H^{2.5}(B_1)} + \frac{\rho'}{\rho} \|\mathbf{h}^2\|_{H^{2.5}(\mathbb{S}^1)} \Big] \\ &\leqslant C\rho^{-1} \|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} \Big[ \rho^{-1} \|\mathbf{h}\|_{H^{5.5}(\mathbb{S}^1)} + \rho' \|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)} \Big]. \end{aligned} \tag{67}$$

### 5.2 *The $H^{2.5}$-decay estimate*

Define

$$I_k(t) = \int_0^t \Big[ \frac{|k|(|k|^2 - 1)}{\rho^3} + (|k| + 1) \frac{\rho'}{\rho} \Big](t') dt' = \int_0^t \frac{|k|(|k|^2 - 1)}{\rho(t')^3} \, dt' + |k| \log \rho(t)\,.$$

Then the use of $I_k(t)$ as the integrating factor in (66) implies that

$$\widehat{\mathbf{h}}_k(t) = e^{-I_k(t)}\widehat{\mathbf{h}}_{0k} + \int_0^t e^{-I_k(t)+I_k(s)}\widehat{R}_k(s)ds$$

Since $d(t) + 2\log\rho(t) = I_2(t) \leqslant I_k(t)$ for all $|k| \geqslant 2$, and $\rho(s) \leqslant \rho(t)$ for all $s \leqslant t$, we find that

$$|\widehat{\mathbf{h}}_k(t)| \leqslant \frac{e^{-d(t)}}{\rho(t)^2}\Big[|\widehat{\mathbf{h}}_{0k}| + \int_0^t \rho(s)^2 e^{d(s)}|\widehat{R}_k(s)|ds\Big] \qquad \forall\, |k| \geqslant 2;$$

thus by Hölder's inequality,

$$\sum_{k\neq 0,\pm 1}(1+|k|^5)|\widehat{\mathbf{h}}_k(t)|^2 \leqslant \frac{Ce^{-2d(t)}}{\rho(t)^4}\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)} + t\int_0^t \rho(s)^4 e^{2d(s)}\|R(s)\|^2_{H^{2.5}(\mathbb{S}^1)}ds\Big]. \quad (68)$$

Since $\rho'(t)$ is bounded, by (67) and interpolation we obtain that

$$\int_0^t \rho(s)^4 e^{2d(s)}\|R(s)\|^2_{H^{2.5}(\mathbb{S}^1)}ds \leqslant C\int_0^t e^{2d(s)}\|\mathbf{h}(s)\|^2_{H^{2.5}(\mathbb{S}^1)}\Big[\|\mathbf{h}(s)\|^2_{H^{5.5}(\mathbb{S}^1)} + \rho(s)^2\rho'(s)^2\|\mathbf{h}(s)\|^2_{H^{3.5}(\mathbb{S}^1)}\Big]ds. \quad (69)$$

5.2.1 *The case that $\rho$ satisfies* (13). In this case, by interpolation (69) implies that

$$\int_0^t \rho(s)^4 e^{2d(s)}\|R(s)\|^2_{H^{2.5}(\mathbb{S}^1)}ds \leqslant C\int_0^t \|\mathbf{h}(s)\|^{16/7}_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}(s)\|^{12/7}_{H^{6}(\mathbb{S}^1)}ds + C\int_0^t \rho(s)^2\rho'(s)^2\|\mathbf{h}(s)\|^{24/7}_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}(s)\|^{4/7}_{H^{5}(\mathbb{S}^1)}ds.$$

Since $\|\mathbf{h}(t)\|_{H^6(\mathbb{S}^1)} + \rho(t)^2 e^{\beta d(t)}\|\mathbf{h}(t)\|_{H^{2.5}(\mathbb{S}^1)} \leqslant |||\mathbf{h}|||_T$, we find that

$$\begin{aligned}\int_0^t \rho(s)^4 e^{2d(s)}\|R(s)\|^2_{H^{2.5}(\mathbb{S}^1)}ds &\leqslant C\Big[\int_0^t \Big(\frac{e^{(2-\frac{16}{7}\beta)d(s)}}{\rho(s)^{32/7}} + \frac{\rho'(s)^2 e^{(2-\frac{24}{7}\beta)d(s)}}{\rho(s)^{34/7}}\Big)ds\Big]|||\mathbf{h}|||^4_T \\ &\leqslant C\Big[\int_0^t \Big(\Big(\frac{1}{\rho(s)^{11/7}} + \frac{\rho'(s)^2}{\rho(s)^{13/7}}\Big)e^{(2-\frac{16}{7}\beta)d(s)}d'(s)\Big)ds\Big]|||\mathbf{h}|||^4_T;\end{aligned}$$

thus by (13), we obtain that

$$\int_0^t \rho(s)^4 e^{2d(s)}\|R(s)\|^2_{H^{2.5}(\mathbb{S}^1)}ds \leqslant C\int_0^t e^{(2-\frac{16}{7}\beta)d(s)}d'(s)ds = \frac{C}{2-\frac{16}{7}\beta}\,e^{(2-\frac{16}{7}\beta)d(s)}\Big|_{s=0}^{s=t} \leqslant C_\beta e^{(2-\frac{16}{7}\beta)d(t)}$$

if $2-\dfrac{16}{7}\beta>0$ or $\beta<\dfrac{7}{8}$. Inequality (68) then implies that

$$\sum_{|k|\geqslant 2}(1+|k|^5)|\widehat{\mathbf{h}}_k(t)|^2 \leqslant C\frac{e^{-2d(t)}}{\rho(t)^4}\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)}+C_\beta t e^{(2-\frac{16}{7}\beta)d(t)}|||\mathbf{h}|||_T^4\Big]$$
$$\leqslant C\frac{e^{-2\beta d(t)}}{\rho(t)^4}\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)}+C_\beta|||\mathbf{h}|||_T^4\Big], \tag{70}$$

where we conclude the last inequality from $\sup\limits_{t>0} t e^{-2\beta d(t)/7}<\infty$ which is a direct consequence of assumption (13) as well.

5.2.2 *The case that $\rho$ satisfies assumption (15).* In this case,

$$d(t)=\int_0^t \frac{6}{\rho(s)^3}ds\leqslant C\int_0^\infty \frac{1}{(1+t)^{3\nu-}}ds<\infty \qquad \forall\, t>0;$$

thus $d$ is bounded. Since

$$\|\mathbf{h}(t)\|_{H^k(\mathbb{S}^1)}+\rho(t)^{2-\frac{1}{2\nu-}}\|\mathbf{h}(t)\|_{H^{2.5}(\mathbb{S}^1)}\leqslant |||\mathbf{h}|||_T,$$

by interpolation (69) implies that

$$\int_0^t \rho(s)^4 e^{2d(s)}\|R(s)\|^2_{H^{2.5}(\mathbb{S}^1)}ds\leqslant C\int_0^t \|\mathbf{h}(s)\|^{2+2\frac{\mathrm{K}-5.5}{\mathrm{K}-2.5}}_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}(s)\|^{\frac{6}{\mathrm{K}-2.5}}_{H^{\mathrm{K}}(\mathbb{S}^1)}ds$$
$$+C\int_0^t \rho(s)^2\rho'(s)^2\|\mathbf{h}(s)\|^{2+2\frac{\mathrm{K}-3.5}{\mathrm{K}-2.5}}_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}(s)\|^{\frac{2}{\mathrm{K}-2.5}}_{H^{\mathrm{K}}(\mathbb{S}^1)}ds$$
$$\leqslant C\Big[\int_0^t\Big(\rho(s)^{\frac{1-4\nu^-}{\nu^-}\frac{2\mathrm{K}-8}{\mathrm{K}-2.5}}+\rho(s)^{\frac{1-4\nu^-}{\nu^-}\frac{2\mathrm{K}-6}{\mathrm{K}-2.5}+2}\rho'(s)^2\Big)ds\Big]|||\mathbf{h}|||_T^4.$$

Note that by the definition of K, $\mathrm{K}>\dfrac{64\nu-21}{16\nu-6}$; thus by (15),

$$\rho(s)^{\frac{1-4\nu^-}{\nu^-}\frac{2\mathrm{K}-8}{\mathrm{K}-2.5}}+\rho(s)^{\frac{1-4\nu^-}{\nu^-}\frac{2\mathrm{K}-6}{\mathrm{K}-2.5}+2}\rho'(s)^2\leqslant C_\sigma(1+t)^{-\sigma}$$

for some $\sigma=\sigma(\nu)>1$. Therefore,

$$\frac{te^{-2d(t)}}{\rho(t)^4}\int_0^t e^{2d(s)}\|R(s)\|^2_{H^{2.5}(\mathbb{S}^1)}ds\leqslant \frac{C_\sigma t}{\rho(t)^4}\int_0^\infty (1+t)^{-\sigma}ds\leqslant \frac{Ct}{\rho(t)^4};$$

and we then conclude from (68) that

$$\frac{\rho(t)^4}{1+t}\sum_{|k|\geqslant 2}(1+|k|^5)|\widehat{\mathbf{h}}_k(t)|^2\leqslant C\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)}+|||\mathbf{h}|||_T^4\Big]. \tag{71}$$

### 5.3 *The mass and the moment of $\Omega(t)$*

For $\mathbf{x} \in \Omega(t)$, we write $\mathbf{x} = (x, y)$. Equation (2) shows that $\int_{\Omega(t)} = \pi\rho(t)^2$; hence,

$$\pi\rho(t)^2 = \frac{1}{2}\oint(xdy - ydx) = \frac{1}{2}\int_0^{2\pi} \big(\rho(t) + \mathbf{h}(\theta, t)\big)^2 d\theta.$$

As a consequence,

$$\begin{aligned}\widehat{\mathbf{h}}_0(t) &= -\frac{1}{2\rho}\widehat{\mathbf{h}^2}_0(t) = -\frac{1}{2\rho}\sum_{k\in\mathbb{Z}}\widehat{\mathbf{h}}_k\widehat{\mathbf{h}}_{-k} \\ &= -\frac{1}{2\rho}\Big[|\widehat{\mathbf{h}}_0|^2 + 2\widehat{\mathbf{h}}_1\widehat{\mathbf{h}}_{-1}\Big] - \frac{1}{2\rho}\sum_{k\neq -1,0,1}\widehat{\mathbf{h}}_k\widehat{\mathbf{h}}_{-k}.\end{aligned} \tag{72}$$

Moreover, by Green's identity and the fact that $\displaystyle\int_{\Gamma(t)} \mathrm{H}n\,dS = 0$, we see that

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)}(x, y)dA &= \int_{\Gamma(t)}(x, y)(u\cdot n)dS = -\int_{\Gamma(t)}(x, y)\frac{\partial p}{\partial n}\,dS \\ &= -\int_{\Gamma(t)}\frac{\partial(x, y)}{\partial n}p\,dS + \int_{\Omega(t)}(x, y)\Delta p\,dx = -\int_{\Gamma(t)}\mathrm{H}n\,dS = 0.\end{aligned}$$

Therefore, the moment of $\Omega(t)$ does not change in time; thus

$$x_0 = \int_{\Omega(t)} x\,dA = \int_0^{2\pi}\int_0^{\rho+\mathbf{h}} r^2\cos\theta\,dr\,d\theta = \frac{1}{3}\int_0^{2\pi}(\rho + \mathbf{h})^3\cos\theta\,d\theta$$

and

$$y_0 = \int_{\Omega(t)} y\,dA = \int_0^{2\pi}\int_0^{\rho+\mathbf{h}} r^2\sin\theta\,dr\,d\theta = \frac{1}{3}\int_0^{2\pi}(\rho + \mathbf{h})^3\sin\theta\,d\theta.$$

Letting $x_0 + iy_0 = r_0e^{i\theta_0}$, we see that

$$\int_0^{2\pi}\big(\rho(t) + \mathbf{h}(\theta, t)\big)^3 e^{\pm i\theta}d\theta = 3r_0e^{\pm i\theta_0};$$

thus

$$\rho^2\widehat{\mathbf{h}}_{\pm 1} + \rho\widehat{\mathbf{h}^2}_{\pm 1} + \frac{1}{3}\widehat{\mathbf{h}^3}_{\pm 1} = 3r_0e^{\pm i\theta_0}.$$

In particular, since $\widehat{fg}_j = \sum_{k\in\mathbb{Z}} \widehat{f}_k \widehat{g}_{j-k}$, we find that

$$
\begin{aligned}
\widehat{\mathbf{h}}_1 &= \frac{3r_0 e^{i\theta_0}}{\rho^2} - \frac{1}{\rho}\sum_{k\in\mathbb{Z}} \widehat{\mathbf{h}}_k \widehat{\mathbf{h}}_{1-k} - \frac{1}{3\rho^2}\sum_{\ell,k\in\mathbb{Z}} \widehat{\mathbf{h}}_{k-\ell}\widehat{\mathbf{h}}_\ell \widehat{\mathbf{h}}_{1-k} \\
&= \frac{3r_0 e^{i\theta_0}}{\rho^2} - \Big[\frac{2\widehat{\mathbf{h}}_0}{\rho} + \frac{\|\mathbf{h}\|^2_{L^2(\mathbb{S}^1)}}{3\rho^2} + \frac{2|\widehat{\mathbf{h}}_0|^2}{3\rho^2}\Big]\widehat{\mathbf{h}}_1 - \Big(\frac{1}{\rho} + \frac{2\widehat{\mathbf{h}}_0 + \widehat{\mathbf{h}}_1 + \widehat{\mathbf{h}}_{-1}}{3\rho^2}\Big)\sum_{k\neq 0,1} \widehat{\mathbf{h}}_k \widehat{\mathbf{h}}_{1-k} \\
&\quad - \frac{1}{3\rho^2}\sum_{|\ell-2|\geqslant 2} \widehat{\mathbf{h}}_{2-\ell}\widehat{\mathbf{h}}_\ell \widehat{\mathbf{h}}_{-1} - \frac{1}{3\rho^2}\sum_{k\neq 0,1,2}\sum_{|\ell-k|\geqslant 2} \widehat{\mathbf{h}}_{k-\ell}\widehat{\mathbf{h}}_\ell \widehat{\mathbf{h}}_{1-k}.
\end{aligned} \tag{73}
$$

By (70) and the Schwarz inequality,

$$
\begin{aligned}
\sum_{k\neq 0,\pm 1} \widehat{\mathbf{h}}_k \widehat{\mathbf{h}}_{\pm 1-k} &\leqslant \frac{C}{\mathfrak{D}(t)}\Big[\|\mathbf{h}_0\|_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||^2_T\Big]\sum_{k\in\mathbb{Z}} \frac{|\widehat{\mathbf{h}}_k|}{1+|k|^{2.5}} \\
&\leqslant \frac{C}{\mathfrak{D}(t)}\Big[\|\mathbf{h}_0\|_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||^2_T\Big]\|\mathbf{h}\|_{L^2(\mathbb{S}^1)}.
\end{aligned} \tag{74}
$$

On the other hand, since $|\widehat{\mathbf{h}}_j| \leqslant \|\mathbf{h}\|_{L^2(\mathbb{S}^1)}$,

$$
\begin{aligned}
\sum_{|\ell-2|\geqslant 2} \widehat{\mathbf{h}}_{2-\ell}\widehat{\mathbf{h}}_\ell \widehat{\mathbf{h}}_{-1} &\leqslant \frac{C}{\mathfrak{D}(t)}\Big[\|\mathbf{h}_0\|_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||^2_T\Big]\sum_{|\ell-2|\geqslant 2} \frac{|\widehat{\mathbf{h}}_\ell||\widehat{\mathbf{h}}_{-1}|}{1+|\ell-2|^{2.5}} \\
&\leqslant \frac{C}{\mathfrak{D}(t)}\Big[\|\mathbf{h}_0\|_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||^2_T\Big]\|\mathbf{h}\|^2_{L^2(\mathbb{S}^1)},
\end{aligned} \tag{75}
$$

and similarly,

$$
\begin{aligned}
&\sum_{k\neq 0,1,2}\sum_{|\ell-k|\geqslant 2} \widehat{\mathbf{h}}_{k-\ell}\widehat{\mathbf{h}}_\ell \widehat{\mathbf{h}}_{1-k} \\
&\qquad \leqslant \frac{C}{\mathfrak{D}(t)^2}\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||^4_T\Big]\sum_{k\neq 0,1,2}\sum_{|\ell-k|\geqslant 2} \frac{|\widehat{\mathbf{h}}_\ell|}{(1+|k-\ell|^{2.5})(1+|k|^{2.5})} \\
&\qquad \leqslant \frac{C}{\mathfrak{D}(t)^2}\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||^4_T\Big]\|\mathbf{h}\|_{L^2(\mathbb{S}^1)}.
\end{aligned} \tag{76}
$$

Moreover, by assumption (39), $|\widehat{\mathbf{h}}_0| \leqslant 2\pi\sigma \ll 1$ and $\|\mathbf{h}\|_{L^2(\mathbb{S}^1)} \leqslant \sqrt{2\pi}\sigma \ll 1$. As a consequence, (73) together with (74), (75) and (76) implies that

$$
|\widehat{\mathbf{h}}_1| \leqslant \frac{C}{\mathfrak{D}(t)}\Big[\|\mathbf{h}_0\|_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||^2_T + |||\mathbf{h}|||^4_T\Big]. \tag{77}
$$

A similar argument also suggests that $|\widehat{\mathbf{h}}_{-1}|$ shares the same upper bound as $|\widehat{\mathbf{h}}_1|$. Therefore, once again using the inequality $|\widehat{\mathbf{h}}_0| \leqslant 2\pi\sigma \ll 1$, (72) together with (74) implies that

$$|\widehat{\mathbf{h}}_0| \leqslant C\rho^{-1}\Big[|\widehat{\mathbf{h}}_1\widehat{\mathbf{h}}_{-1}| + \frac{1}{\mathfrak{D}(t)}\Big(\|\mathbf{h}_0\|_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||_T^2\Big)\|\mathbf{h}\|_{L^2(\mathbb{S}^1)}\Big]$$
$$\leqslant \frac{C}{\mathfrak{D}(t)}\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||_T^2 + |||\mathbf{h}|||_T^8\Big]. \tag{78}$$

Combining (70) (or (71)), (77) and (78), we conclude that

$$\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} = \Big[\sum_{k\in\mathbb{Z}}(1+|k|^5)|\widehat{\mathbf{h}}_k|^2\Big]^{\frac{1}{2}} \leqslant \frac{C}{\mathfrak{D}(t)}\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||_T^2 + |||\mathbf{h}|||_T^8\Big]. \tag{79}$$

Furthermore, since

$$\int_0^\infty \frac{\rho(t)^{-3} + \rho(t)^{-1}\rho'(t)^2\big]}{\mathfrak{D}(t)^2}\,dt < \infty$$

for $\mathfrak{D}$ in both the slow and fast injection cases, (61) implies that

$$\int_0^\infty \rho(t)^{-3}\|\mathbf{h}(t)\|^2_{H^{\mathrm{K}+1.5}(\mathbb{S}^1)}dt$$
$$\leqslant C\int_0^\infty \rho(t)^3\|v(t)\|^2_{H^{k-1}(B_1)}dt + C\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)} + |||\mathbf{h}|||_T^4 + |||\mathbf{h}|||_T^{16}\Big]. \tag{80}$$

## 6. Energy and bounds for the total norm for positive surface tension

### 6.1 *Energy estimates*

Recall that $\bar\partial = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ denotes the tangential derivative. Tangentially differentiating (28a) $\ell$-times ($\ell = 0,1,2,3,4,5$) and then testing the resulting equation against $\mathrm{J}\bar\partial^\ell v$, we find that

$$\int_{B_1}\mathrm{J}|\bar\partial^\ell \mathbf{v}|^2dx + \int_{B_1}\mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell\mathbf{q},_j\,\bar\partial^\ell\mathbf{v}^i\,dx = -\sum_{k=0}^{\ell-1}\binom{\ell}{k}\int_{B_1}\mathrm{J}\bar\partial^{\ell-k}\mathrm{A}_i^j\,\bar\partial^k\mathbf{q},_j\,\bar\partial^\ell\mathbf{v}^i\,dx.$$

Writing the second integral on the left-hand side as

$$\int_{B_1}\mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell\mathbf{q},_j\,\bar\partial^\ell\mathbf{v}^i\,dx = \int_{B_1}\mathrm{J}\mathrm{A}_i^j\,(\bar\partial^\ell\mathbf{q}),_j\,\bar\partial^\ell\mathbf{v}^i\,dx + \int_{B_1}\mathrm{J}\mathrm{A}_i^j\,[\bar\partial^\ell\mathbf{q},_j - (\bar\partial^\ell\mathbf{q}),_j\,]\bar\partial^\ell\mathbf{v}^i\,dx,$$

then integrating by parts in $x_j$ leads to

$$\int_{B_1}\mathrm{J}|\bar\partial^\ell\mathbf{v}|^2dx + \int_{\mathbb{S}^1}\mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell\mathbf{q}\bar\partial^\ell\mathbf{v}^i\mathrm{N}_j\,dS = -\sum_{k=0}^{\ell-1}\binom{\ell}{k}\int_{B_1}\mathrm{J}\bar\partial^{\ell-k}\mathrm{A}_i^j\,\bar\partial^k\mathbf{q},_j\,\bar\partial^\ell\mathbf{v}^i\,dx$$
$$+\int_{B_1}\mathrm{J}\mathrm{A}_i^j\Big[[(\bar\partial^\ell\mathbf{q}),_j - \bar\partial^\ell\mathbf{q},_j\,]\bar\partial^\ell\mathbf{v}^i + \bar\partial^\ell\mathbf{q}(\bar\partial^\ell\mathbf{v}^i),_j\Big]dx. \tag{81}$$

Using the Kronecker delta symbol $\delta_{0\ell}$, which vanishes for all $\ell \neq 0$, by (44), (45) and (57) together with the continuous embedding $H^{\frac{p}{p-2}}(B_1) \hookrightarrow L^p(B_1)$ we find that

$$\Big|\int_{B_1} \mathrm{J}\mathrm{A}_i^j \Big[\big[\bar\partial^\ell \mathbf{q},_j - (\bar\partial^\ell \mathbf{q}),_j\big]\bar\partial^\ell \mathbf{v}^i + \bar\partial^\ell \mathbf{q}\big[(\bar\partial^\ell \mathbf{v}^i),_j - \bar\partial^\ell \mathbf{v}^i,_j\big]\Big] dx\Big| \\ \leqslant C\rho^2(1-\delta_{0\ell})\|\mathbf{v}\|_{H^{\ell-1}(B_1)}\|\mathbf{v}\|_{H^\ell(B_1)},$$

as well as for $0 \leqslant k \leqslant \ell - 2$,

$$\Big|\int_{B_1} \mathrm{J}\bar\partial^{\ell-k}\mathrm{A}_i^j\,\bar\partial^k \mathbf{q},_j\,\bar\partial^\ell \mathbf{v}^i\,dx\Big| \leqslant C\rho^2\|D\mathrm{A}\|_{W^{\ell-k-1,4}(B_1)}\|\nabla\mathbf{q}\|_{W^{k,4}(B_1)}\|\mathbf{v}\|_{H^\ell(B_1)} \\ \leqslant C\rho\|\mathbf{h}\|_{H^{\ell-k+1}(\mathbb{S}^1)}\|\mathbf{v}\|_{H^{k+0.5}(B_1)}\|\mathbf{v}\|_{H^\ell(B_1)}$$

and for $k = \ell - 1$,

$$\Big|\int_{B_1} \mathrm{J}\bar\partial^{\ell-k}\mathrm{A}_i^j\,\bar\partial^k \mathbf{q},_j\,\bar\partial^\ell \mathbf{v}^i\,dx\Big| \leqslant C\rho^3\|D\mathrm{A}\|_{L^\infty(B_1)}\|\mathbf{v}\|_{H^{\ell-1}(B_1)}\|\mathbf{v}\|_{H^\ell(B_1)} \\ \leqslant C\rho\|\mathbf{h}\|_{H^3(\mathbb{S}^1)}\|\mathbf{v}\|_{H^{\ell-1}(B_1)}\|\mathbf{v}\|_{H^\ell(B_1)}.$$

Therefore, (55) suggests that

$$\int_{B_1} \mathrm{J}|\bar\partial^\ell \mathbf{v}|^2 dx + \int_{\mathbb{S}^1} \mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell \mathbf{q}\bar\partial^\ell \mathbf{v}^i \mathrm{N}_j\,dS \\ \leqslant \int_{B_1} \mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell \mathbf{q}\bar\partial^\ell \mathbf{v}^i,_j\,dx + C(1-\delta_{0\ell})\rho^2\|\mathbf{v}\|_{H^{\ell-1}(B_1)}\|\mathbf{v}\|_{H^\ell(B_1)} \\ + C(1-\delta_{0\ell})\|\mathbf{h}\|_{H^{\ell+1}(\mathbb{S}^1)}\|\mathbf{v}\|_{H^\ell(B_1)}\|\mathbf{v}\|_{H^1(B_1)}. \quad (82)$$

By (28b), $\mathrm{A}_i^j\,\bar\partial^\ell \mathbf{v}^i,_j = -\sum_{k=0}^{\ell-1}\binom{\ell}{k}\bar\partial^{\ell-k}\mathrm{A}_i^j\,\bar\partial^k \mathbf{v}^i,_j$; thus for $\ell \geqslant 1$,

$$\int_{B_1} \mathrm{J}\bar\partial^\ell \mathbf{q}\mathrm{A}_i^j\,\bar\partial^\ell \mathbf{v}^i,_j\,dx \leqslant C\rho^2(1-\delta_{0\ell})\Big[\|\nabla\mathbf{q}\|_{H^{\ell-1}(B_1)}\|D\mathrm{A}\|_{L^\infty(B_1)}\|\mathbf{v}\|_{H^\ell(B_1)} \\ + \|\nabla\mathbf{q}\|_{H^{\ell-0.5}(B_1)}\sum_{k=0}^{\ell-2}\|D\mathrm{A}\|_{H^{\ell-k-0.5}(B_1)}\|\mathbf{v}\|_{H^{k+1}(B_1)}\Big] \\ \leqslant C\rho(1-\delta_{0\ell})\Big[\|\mathbf{h}\|_{H^{\ell+1}(\mathbb{S}^1)}\|\mathbf{v}\|_{H^\ell(B_1)}\|\mathbf{v}\|_{H^1(B_1)} + \|\mathbf{h}\|_{H^3(\mathbb{S}^1)}\|\mathbf{v}\|_{H^{\ell-1}(B_1)}\|\mathbf{v}\|_{H^\ell(B_1)}\Big].$$

We now focus on the second integral of the left-hand side of (82). By identity (32) and the boundary condition (28c),

$$\begin{aligned}\int_{\mathbb{S}^1} \mathrm{J}\mathrm{A}_i^j \bar\partial^\ell \mathbf{q} \bar\partial^\ell \mathbf{v}^i \mathrm{N}_j \, dS &= \int_{\mathbb{S}^1} (\rho + \mathbf{h})(\bar\partial^\ell \mathbf{v} \cdot \mathfrak{n}) \bar\partial^\ell \mathbf{q} dS \\ &= \int_{\mathbb{S}^1} (\rho + \mathbf{h}) \Big[ \bar\partial^\ell (\mathbf{v} \cdot \mathfrak{n}) - \sum_{k=0}^{\ell-1} \binom{\ell}{k} \bar\partial^k \mathbf{v} \cdot \bar\partial^{\ell-k} \mathfrak{n} \Big] \bar\partial^\ell \mathbf{q} \, dS \\ &= \int_{\mathbb{S}^1} (\rho + \mathbf{h}) \bar\partial^\ell (\mathbf{v} \cdot \mathfrak{n}) \bar\partial^\ell \Big[ - \mathbf{J}_\mathbf{h}^{-3} \big[ (\rho + \mathbf{h}) \mathbf{h}_{\theta\theta} \big] + \rho' \mathbf{h} \Big] dS \\ &\quad + \int_{\mathbb{S}^1} (\rho + \mathbf{h}) \bar\partial^\ell (\mathbf{v} \cdot \mathfrak{n}) \bar\partial^\ell \Big[ \mathbf{J}_\mathbf{h}^{-3} \big( \mathbf{J}_\mathbf{h}^2 + \mathbf{h}_\theta^2 \big) - \rho^{-1} + \bar{\bar{\mathbf{q}}} \Big] dS \\ &\quad - \sum_{k=0}^{\ell-1} \binom{\ell}{k} \int_{\mathbb{S}^1} (\rho + \mathbf{h}) \bar\partial^k \mathbf{v} \cdot \bar\partial^{\ell-k} \mathfrak{n} \bar\partial^\ell \mathbf{q} dS.\end{aligned}$$

By employing the $H^{0.5}(\mathbb{S}^1)$-$H^{-0.5}(\mathbb{S}^1)$ duality pairing,

$$\begin{aligned}\Big| \sum_{k=0}^{\ell-1} \int_{\mathbb{S}^1} (\rho + \mathbf{h}) \bar\partial^k \mathbf{v} \cdot \bar\partial^{\ell-k} \mathfrak{n} \bar\partial^\ell \mathbf{q} dS \Big| &\leqslant C(1 - \delta_{0\ell}) \rho \Big[ \|\mathbf{v}\|_{L^4(\mathbb{S}^1)} \|\bar\partial^\ell \mathfrak{n}\|_{L^2(\mathbb{S}^1)} \|\bar\partial^\ell \mathbf{q}\|_{L^4(\mathbb{S}^1)} \\ &\quad + (1 - \delta_{0\ell} - \delta_{1\ell}) \|\bar\partial^{\ell-1} \mathbf{v}\|_{H^{-0.5}(\mathbb{S}^1)} \|\bar\partial^\ell \mathbf{q}\|_{H^{0.5}(\mathbb{S}^1)} \|\bar\partial \mathfrak{n}\|_{H^1(\mathbb{S}^1)} \\ &\quad + (1 - \delta_{0\ell} - \delta_{1\ell} - \delta_{2\ell}) \sum_{k=1}^{\ell-2} \|\bar\partial^k \mathbf{v}\|_{H^{-0.5}(\mathbb{S}^1)} \|\bar\partial^\ell \mathbf{q}\|_{H^{0.5}(\mathbb{S}^1)} \|\bar\partial^{\ell-k} \mathfrak{n}\|_{H^1(\mathbb{S}^1)} \Big] \\ &\leqslant C\rho^2 \Big[ \|\mathbf{v}\|_{H^\ell(B_1)} \|\mathbf{v}\|_{H^1(B_1)} \big[ 1 + \rho^{-1} \|\mathbf{h}\|_{H^{\ell+1}(\mathbb{S}^1)} \big] \\ &\quad + \|\mathbf{v}\|_{H^{\ell-1}(B_1)} \|\mathbf{v}\|_{H^\ell(B_1)} \big[ 1 + \rho^{-1} \|\mathbf{h}\|_{H^4(\mathbb{S}^1)} \big] \Big]\end{aligned}$$

and by (48),

$$\begin{aligned}\Big| \int_{\mathbb{S}^1} (\rho + \mathbf{h}) \bar\partial^\ell (\mathbf{v} \cdot \mathfrak{n}) \bar\partial^\ell \bar{\bar{\mathbf{q}}} dS \Big| &\leqslant C\rho \|\mathbf{v} \cdot \mathfrak{n}\|_{H^{\ell-0.5}(\mathbb{S}^1)} \|\bar{\bar{\mathbf{q}}}\|_{H^{\ell+0.5}(\mathbb{S}^1)} \\ &\leqslant C\rho' \|\mathbf{v} \cdot \mathfrak{n}\|_{H^{\ell-0.5}(\mathbb{S}^1)} \|\mathbf{h}\|_{H^2(\mathbb{S}^1)} \|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)}.\end{aligned}$$

Moreover, since

$$\mathbf{J}_\mathbf{h}^{-1} - \rho^{-1} = - \frac{2\rho \mathbf{h} + \mathbf{h}^2 + \mathbf{h}_\theta^2}{\mathbf{J}_\mathbf{h} \rho (\rho + \mathbf{J}_\mathbf{h})}$$

and

$$\frac{2(\rho + \mathbf{h})}{\mathbf{J}_\mathbf{h}(\rho + \mathbf{J}_\mathbf{h})} - \frac{1}{\rho} = - \Big[ \frac{\mathbf{h}(2\rho + \mathbf{h}) + \mathbf{h}_\theta^2}{\mathbf{J}_\mathbf{h}(\rho + \mathbf{J}_\mathbf{h})^2} + \frac{\mathbf{h}^2 + \mathbf{h}_\theta^2}{\mathbf{J}_\mathbf{h} \rho (\rho + \mathbf{J}_\mathbf{h})} \Big],$$

we find that

$$
\begin{aligned}
&-\int_{\mathbb{S}^1}(\rho+\mathbf{h})\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^\ell\big[\mathbf{J}_\mathbf{h}^{-1}-\rho^{-1}\big]dS\\
&\qquad=\int_{\mathbb{S}^1}(\rho+\mathbf{h})\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^\ell\Big[\frac{2\mathbf{h}}{\mathbf{J}_\mathbf{h}(\rho+\mathbf{J}_\mathbf{h})}+\frac{\mathbf{h}^2+\mathbf{h}_\theta^2}{\mathbf{J}_\mathbf{h}\rho(\rho+\mathbf{J}_\mathbf{h})}\Big]dS\\
&\qquad\leqslant\int_{\mathbb{S}^1}\frac{2(\rho+\mathbf{h})}{\mathbf{J}_\mathbf{h}(\rho+\mathbf{J}_\mathbf{h})}\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^\ell\mathbf{h}dS+C\rho^{-2}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{\ell+1.5}(\mathbb{S}^1)}\|\mathbf{v}\|_{H^\ell(B_1)}\\
&\qquad\leqslant\frac{1}{\rho}\int_{\mathbb{S}^1}\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^\ell\mathbf{h}\,dS+C\rho^{-2}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{\ell+1.5}(\mathbb{S}^1)}\|\mathbf{v}\cdot\mathfrak{N}\|_{H^{\ell-0.5}(\mathbb{S}^1)}\\
&\qquad\qquad+C\rho^{-2}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{\ell+1.5}(\mathbb{S}^1)}\|\mathbf{v}\|_{H^\ell(B_1)}.
\end{aligned}
$$

Finally, because of the identity

$$
\frac{(\rho+\mathbf{h})^2}{\mathbf{J}_\mathbf{h}^3}-\frac{1}{\rho}=-\frac{(\rho+\mathbf{h})^4(2\rho+\mathbf{h})\mathbf{h}+3(\rho+\mathbf{h})^4\mathbf{h}_\theta^2+3(\rho+\mathbf{h})^2\mathbf{h}_\theta^4+\mathbf{h}_\theta^6}{\mathbf{J}_\mathbf{h}^3\rho\big[\mathbf{J}_\mathbf{h}^3+\rho(\rho+\mathbf{h})^2\big]}
$$

by the evolution equation (31) we obtain that

$$
\begin{aligned}
&\int_{\mathbb{S}^1}(\rho+\mathbf{h})\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^\ell\Big[-\mathbf{J}_\mathbf{h}^{-3}(\rho+\mathbf{h})\mathbf{h}_{\theta\theta}+\rho'\mathbf{h}\Big]dS\\
&\quad=-\int_{\mathbb{S}^1}\mathbf{J}_\mathbf{h}^{-3}(\rho+\mathbf{h})^2\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^{\ell+2}\mathbf{h}\,dS+\int_{\mathbb{S}^1}(\rho+\mathbf{h})\rho'\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^\ell\mathbf{h}\,dS\\
&\qquad-\sum_{k=0}^{\ell-1}\binom{\ell}{k}\int_{\mathbb{S}^1}(\rho+\mathbf{h})\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^{\ell-k}\big[\mathbf{J}_\mathbf{h}^{-3}(\rho+h)\big]\bar\partial^{k+1}\mathbf{h}\,dS\\
&\quad\geqslant\frac{1}{2\rho}\frac{d}{dt}\|\bar\partial^{\ell+1}\mathbf{h}\|_{L^2(\mathbb{S}^1)}^2+\frac{1}{2}\rho^{-3}\rho'\|\bar\partial^{\ell+1}\mathbf{h}\|_{L^2(\mathbb{S}^1)}^2\\
&\qquad+\int_{\mathbb{S}^1}(\rho+\mathbf{h})\rho'\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^\ell\mathbf{h}\,dS-C\rho^{-4}\rho'\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}^2\|\mathbf{h}\|_{H^\ell(\mathbb{S}^1)}^2\\
&\qquad-C\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{v}\cdot\mathfrak{N}\|_{H^{\ell-0.5}(\mathbb{S}^1)}\Big[\rho^{-2}\|\mathbf{h}\|_{H^{\ell+2.5}(\mathbb{S}^1)}+\rho'\|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)}\Big]\\
&\qquad-C\rho^{-1}\|\mathbf{v}\cdot\mathfrak{N}\|_{H^{\ell-0.5}(\mathbb{S}^1)}\Big[\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)}+\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}^2\|\mathbf{h}\|_{H^{\ell+1.5}(\mathbb{S}^1)}\Big].
\end{aligned}
$$

Since $\operatorname{div}(\mathrm{JA}^{\mathrm{T}}\tilde v)=0$, by the normal trace theorem we find that

$$
\begin{aligned}
\|\mathbf{v}\cdot\mathfrak{N}\|_{H^{-0.5}(\mathbb{S}^1)}&=\|(\rho+\mathbf{h})^{-1}\mathrm{JA}^{\mathrm{T}}\mathbf{v}\cdot\mathrm{N}\|_{H^{-0.5}(\mathbb{S}^1)}\\
&\leqslant C\Big[\|(\rho+\mathbf{h})^{-1}\mathrm{JA}^{\mathrm{T}}\mathbf{v}\|_{L^2(B_1)}+\|\nabla(\rho+\mathbf{h})^{-1}\mathrm{JA}^{\mathrm{T}}\mathbf{v}\|_{L^2(B_1)}\Big]\leqslant C\|\mathbf{v}\|_{L^2(B_1)}
\end{aligned}
$$

and similarly, for $0\leqslant\ell\leqslant\mathrm{K}-1$,

$$
\|\mathbf{v}\cdot\mathfrak{N}\|_{H^{\ell-0.5}(\mathbb{S}^1)}\leqslant C\big(1+\rho^{-1}|||\mathbf{h}|||_T\big)\|\mathbf{v}\|_{H^\ell(B_1)}.
$$

Therefore, (82) and Young's inequality imply that

$$
\begin{aligned}
&\rho^3\|\bar\partial^\ell \mathbf{v}\|^2_{L^2(B_1)} + \frac{d}{dt}\|\bar\partial^{\ell+1}\mathbf{h}\|^2_{L^2(\mathbb{S}^1)} + 2\int_{\mathbb{S}^1}\rho\rho'(\rho+\mathbf{h})\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^\ell\mathbf{h}dS \\
&\leqslant 2\int_{\mathbb{S}^1}\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{N})\bar\partial^\ell\mathbf{h}dS + C(1-\delta_{0\ell})\rho^3\|\mathbf{v}\|^2_{H^{\ell-1}(B_1)} + \delta\rho^3\|\mathbf{v}\|^2_{H^\ell(B_1)} \\
&\qquad + C_\delta\rho^{-1}\rho'^2\|\mathbf{h}\|^2_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}\|^2_{H^{\ell+1}(\mathbb{S}^1)} + C_\delta\rho^{-5}\|\mathbf{h}\|^2_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}\|^2_{H^{\ell+2.5}(\mathbb{S}^1)} \\
&\qquad\qquad + C\rho^{-3}\rho'\|\mathbf{h}\|^2_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|^2_{H^\ell(\mathbb{S}^1)}. \qquad (83)
\end{aligned}
$$

6.1.1 *The case* $\ell = 0$. Using the $H^{0.5}(\mathbb{S}^1)$-$H^{-0.5}(\mathbb{S}^1)$ duality pairing,

$$
\begin{aligned}
\Big|\int_{\mathbb{S}^1}(\rho\rho'(\rho+\mathbf{h})(\mathbf{v}\cdot\mathfrak{N})\mathbf{h}dS\Big| + \Big|\int_{\mathbb{S}^1}(\mathbf{v}\cdot\mathfrak{N})\mathbf{h}dS\Big| &\leqslant C(\rho^2\rho'+1)\|\mathbf{v}\|_{L^2(B_1)}\|\mathbf{h}\|_{H^{0.5}(\mathbb{S}^1)} \\
&\leqslant C_\delta\big(\rho\rho'^2+\rho^{-3}\big)\|\mathbf{h}\|^2_{H^{2.5}(\mathbb{S}^1)} + \delta\rho^3\|\mathbf{v}\|^2_{L^2(B_1)}.
\end{aligned}
$$

Since

$$
\int_0^\infty\Big[\frac{1}{\rho(s)^3\mathfrak{D}(s)^2} + \frac{\rho'(s)^2}{\rho(s)\mathfrak{D}(s)^4}\Big]ds < \infty\,,
$$

choosing $\delta > 0$ small enough and integrating in time of (83) over the time interval $(0,t)$, by (79) we find that

$$
\begin{aligned}
&\|\bar\partial\mathbf{h}(t)\|^2_{L^2(\mathbb{S}^1)} + \int_0^t\rho(s)^2\|\mathbf{v}(s)\|^2_{L^2(B_1)}ds \\
&\qquad\leqslant C\int_0^t\rho(s)\rho'(s)^2\|\mathbf{h}(s)\|^2_{H^{2.5}(\mathbb{S}^1)}ds + C\Big[\|\mathbf{h}_0\|^2_{H^{2.5}(\mathbb{S}^1)} + |\!|\!|\mathbf{h}|\!|\!|^4_T + |\!|\!|\mathbf{h}|\!|\!|^{16}_T\Big].
\end{aligned}
$$

Define

$$
N(\|\mathbf{h}_0\|_{H^\mathrm{K}(\mathbb{S}^1)}, |\!|\!|\mathbf{h}|\!|\!|_T) \equiv \|\mathbf{h}_0\|^2_{H^\mathrm{K}(\mathbb{S}^1)} + |\!|\!|\mathbf{h}|\!|\!|^4_T\,\mathcal{P}\big(|\!|\!|\mathbf{h}|\!|\!|^2_T\big) \qquad (84)
$$

for some polynomial function $\mathcal{P}$. If (13) is satisfied, due to the exponential decay of $\|\mathbf{h}\|_{H^{2.5}(\mathbb{S})}$, it is easy to see that

$$
\int_0^t\rho(s)\rho'(s)^2\|\mathbf{h}(s)\|^2_{H^{2.5}(\mathbb{S}^1)}ds \leqslant C N(\|\mathbf{h}_0\|_{H^\mathrm{K}(\mathbb{S}^1)}, |\!|\!|\mathbf{h}|\!|\!|_T).
$$

Now suppose that (15) is satisfied. Then (19) and (79) imply that

$$
\begin{aligned}
\int_0^t\rho(s)\rho'(s)^2\|\mathbf{h}(s)\|^2_{H^{2.5}(\mathbb{S}^1)}ds &\leqslant C\Big[\int_0^\infty\frac{\rho(s)^{1/\nu^-}}{(1+s)^2\rho(s)}ds\Big]N(\|\mathbf{h}_0\|_{H^\mathrm{K}(\mathbb{S}^1)}, |\!|\!|\mathbf{h}|\!|\!|_T) \\
&\leqslant C\Big[\int_0^\infty\frac{(1+t)^{\frac{\nu^+}{\nu^-}}}{(1+t)^{2+\nu^-}}\,ds\Big]N(\|\mathbf{h}_0\|_{H^\mathrm{K}(\mathbb{S}^1)}, |\!|\!|\mathbf{h}|\!|\!|_T) \leqslant CN(\|\mathbf{h}_0\|_{H^\mathrm{K}(\mathbb{S}^1)}, |\!|\!|\mathbf{h}|\!|\!|_T);
\end{aligned}
$$

thus in either case,

$$
\|\bar\partial\mathbf{h}(t)\|^2_{L^2(\mathbb{S}^1)} + \int_0^t\rho(s)^3\|\mathbf{v}(s)\|^2_{L^2(B_1)}ds \leqslant CN(\|\mathbf{h}_0\|_{H^\mathrm{K}(\mathbb{S}^1)}, |\!|\!|\mathbf{h}|\!|\!|_T). \qquad (85)
$$

6.1.2 *The case* $1 \leqslant \ell \leqslant \mathrm{K}-1$. Define $\widetilde{\mathbf{h}} = \mathbf{h} + \dfrac{\mathbf{h}^2}{2\rho}$. Then

$$\frac{\rho+\mathbf{h}}{\rho}\Big[\mathbf{v}\cdot\mathfrak{n} - \frac{\rho'\mathbf{h}}{\rho+\mathbf{h}}\Big] = \frac{\big[(\rho+\mathbf{h})^2\big]_t}{2\rho} = \frac{\big[\rho^2+2\rho\widetilde{\mathbf{h}}\big]_t}{2\rho} = \widetilde{\mathbf{h}}_t + \frac{\rho'}{\rho}\widetilde{\mathbf{h}} + \rho'$$

or equivalently,

$$\frac{\rho+\mathbf{h}}{\rho}\,\mathbf{v}\cdot\mathfrak{n} = \widetilde{\mathbf{h}}_t + \frac{2\rho'}{\rho}\widetilde{\mathbf{h}} + \rho' - \frac{\rho'\mathbf{h}^2}{2\rho^2}\,.$$

Therefore, since $\|\mathbf{v}\|_{H^{1.5}(B_1)} \leqslant C\rho^{-1}\Big[\|\mathbf{h}\|_{H^4(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\Big]$ by (55), we find that

$$\begin{aligned}
&\int_{\mathbb{S}^1} \rho\rho'(\rho+\mathbf{h})\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{n})\bar\partial^\ell\mathbf{h}\,dS \\
&\quad= \int_{\mathbb{S}^1} \rho\rho'(\rho+\mathbf{h})\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{n})\bar\partial^\ell\widetilde{\mathbf{h}}\,dS - \frac12\int_{\mathbb{S}^1}\rho'(\rho+\mathbf{h})\bar\partial^\ell(\mathbf{v}\cdot\mathfrak{n})\bar\partial^\ell(\mathbf{h}^2)\,dS \\
&\quad\geqslant \int_{\mathbb{S}^1}\rho\rho'\bar\partial^\ell\big[(\rho+\mathbf{h})(\mathbf{v}\cdot\mathfrak{n})\big]\bar\partial^\ell\widetilde{\mathbf{h}}\,dS - \sum_{k=0}^{\ell-1}\binom{\ell}{k}\int_{\mathbb{S}^1}\rho\rho'\bar\partial^{\ell-k}\mathbf{h}\bar\partial^k(\mathbf{v}\cdot\mathfrak{n})\bar\partial^\ell\widetilde{\mathbf{h}}\,dS \\
&\qquad - C\rho\rho'\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{\ell+1}(\mathbb{S}^1)}\|\mathbf{v}\|_{H^\ell(B_1)} \\
&\quad\geqslant \int_{\mathbb{S}^1}\rho^2\rho'\bar\partial^\ell\Big(\widetilde{\mathbf{h}}_t + \frac{2\rho'}{\rho}\widetilde{\mathbf{h}}\Big)\bar\partial^\ell\widetilde{\mathbf{h}}\,dS - C_\delta\rho^{-1}\rho'^2\|\mathbf{h}\|^2_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}\|^2_{H^{\ell+1}(\mathbb{S}^1)} \\
&\qquad - \delta\rho^3\|\mathbf{v}\|^2_{H^\ell(B_1)} - C\rho'\Big[\|\mathbf{h}\|_{H^4(\mathbb{S}^1)} + \rho'\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\Big]\|\bar\partial^\ell\mathbf{h}\|^2_{L^2(\mathbb{S}^1)}.
\end{aligned}\tag{86}$$

Let us focus on the last term of (86) first. Since $\rho\sqrt{\rho'}\|\mathbf{h}\|_{H^5(\mathbb{S}^1)} \leqslant |||\mathbf{h}|||_T$,

$$\begin{aligned}
&\int_0^\infty\Big[\rho'\|\mathbf{h}(s)\|_{H^4(\mathbb{S}^1)} + \rho'(s)^2\|\mathbf{h}(s)\|_{H^2(\mathbb{S}^1)}\Big]\|\bar\partial^\ell\mathbf{h}\|^2_{L^2(\mathbb{S}^1)}ds \\
&\quad\leqslant \int_0^\infty\Big[\rho'\|\mathbf{h}(s)\|_{H^4(\mathbb{S}^1)} + \rho'(s)^2\|\mathbf{h}(s)\|_{H^2(\mathbb{S}^1)}\Big]\|\bar\partial^\ell\widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)}ds \\
&\qquad + C\int_0^\infty\frac{1}{\rho^2}\Big[\rho'\|\mathbf{h}(s)\|_{H^4(\mathbb{S}^1)} + \rho'(s)^2\|\mathbf{h}(s)\|_{H^2(\mathbb{S}^1)}\Big]\|\bar\partial^\ell(\mathbf{h}^2)\|^2_{L^2(\mathbb{S}^1)}ds \\
&\quad\leqslant C\Big[\int_0^\infty\frac{\|\mathbf{h}(s)\|_{H^4(\mathbb{S}^1)} + \rho'(s)\|\mathbf{h}(s)\|_{H^2(\mathbb{S}^1)}}{\rho(s)^2}ds\Big]|||\mathbf{h}|||^2_T \\
&\qquad + C\,N(\|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)}, |||\mathbf{h}|||_T)
\end{aligned}\tag{87}$$

By interpolation,

$$\|\mathbf{h}(t)\|_{H^4(\mathbb{S}^1)} \leqslant C\,\|\mathbf{h}\|^{\frac{\mathrm{K}-4}{\mathrm{K}-2.5}}_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}\|^{\frac{1.5}{\mathrm{K}-2.5}}_{H^{\mathrm{K}}(\mathbb{S}^1)} \leqslant C\,\mathfrak{D}(t)^{-(\frac{\mathrm{K}-4}{\mathrm{K}-2.5})}|||\mathbf{h}|||_T.$$

If (13) is satisfied, due to the exponential decay it is easy to see that

$$\int_0^\infty\frac{\|\mathbf{h}(s)\|_{H^4(\mathbb{S}^1)} + \rho'(s)\|\mathbf{h}(s)\|_{H^2(\mathbb{S}^1)}}{\rho(s)^2}ds \leqslant C\,|||\mathbf{h}|||_T.\tag{88}$$

Suppose that (15) is satisfied. Then

$$\int_0^\infty \frac{\|\mathbf{h}(s)\|_{H^4(\mathbb{S}^1)} + \rho'(s)\|\mathbf{h}(s)\|_{H^2(\mathbb{S}^1)}}{\rho(s)^2} ds$$
$$\leqslant C\Big[\int_0^\infty \Big(\frac{1}{\rho(s)^{2+(2-\frac{1}{2\nu^-})(\frac{\mathrm{K}-4}{\mathrm{K}-2.5})}} + \frac{1}{(1+s)\rho(s)^{3-\frac{1}{2\nu^-}}}\Big) ds\Big] |||\mathbf{h}|||_T$$
$$\leqslant C\Big[\int_0^\infty \frac{1}{(1+s)^{2\nu^- + (2\nu^- - \frac{1}{2})(\frac{\mathrm{K}-4}{\mathrm{K}-2.5})}} ds\Big] |||\mathbf{h}|||_T + C\,|||\mathbf{h}|||_T.$$

By the definition of K, the exponent of the integrand $2\nu^- + \big(2\nu^- - \frac{1}{2}\big)\big(\frac{\mathrm{K}-4}{\mathrm{K}-2.5}\big) > 1$ if $\nu^- > 3/8$, so (88) is still valid if (15) holds. Therefore, (87) implies that

$$\int_0^\infty \Big[\rho'\|\mathbf{h}(s)\|_{H^4(\mathbb{S}^1)} + \rho'(s)^2\|\mathbf{h}(s)\|_{H^2(\mathbb{S}^1)}\Big]\|\bar\partial^\ell \mathbf{h}\|^2_{L^2(\mathbb{S}^1)} ds$$
$$\leqslant C\,|||\mathbf{h}|||_T^3 + C\,N(\|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)}, |||\mathbf{h}|||_T) \leqslant C_{\delta_1} N(\|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)}, |||\mathbf{h}|||_T) + \delta_1 |||\mathbf{h}|||_T^2.$$

Since

$$\int_{\mathbb{S}^1} \rho^2\rho'\bar\partial^\ell\Big(\widetilde{\mathbf{h}}_t + \frac{2\rho'}{\rho}\widetilde{\mathbf{h}}\Big)\bar\partial^\ell\widetilde{\mathbf{h}}\,dS$$
$$= \frac{1}{2}\frac{d}{dt}\Big[\rho^2\rho'\|\bar\partial^\ell\widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)}\Big] + \rho\rho'^2\|\bar\partial^\ell\widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)} - \frac{1}{2}\rho^2\rho''\|\bar\partial^\ell\widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)},$$

the combination of (83), (86), (87) and (88) suggests that

$$\|\bar\partial^{\ell+1}\mathbf{h}\|^2_{L^2(\mathbb{S}^1)} + \rho^2\rho'\|\bar\partial^\ell\widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)} + c\int_0^t \rho^3\|\bar\partial^\ell\mathbf{v}\|^2_{L^2(B_1)}ds - \int_0^t \rho^2\rho''\|\bar\partial^\ell\widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)}ds$$
$$\leqslant \|\bar\partial^\ell\mathbf{h}\|^2_{L^2(\mathbb{S}^1)} + C_{\delta,\delta_1}N(\|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)}, |||\mathbf{h}|||_T)$$
$$+ C_\delta\int_0^t \rho^3\|\mathbf{v}\|^2_{H^{\ell-1}(B_1)}ds + \delta\int_0^t \rho^3\|\mathbf{v}\|^2_{H^\ell(B_1)}ds + \delta_1|||\mathbf{h}|||_T^2$$

for some constant $0 < c < 1$.

On the other hand, for $0 \leqslant \ell \leqslant \mathrm{K}-1$, assumption (39) and condition (13) or (15) imply that

$$\rho^2\rho'\|\bar\partial^\ell\mathbf{h}\|^2_{L^2(\mathbb{S}^1)} \leqslant 2\rho^2\rho'\Big[\|\bar\partial^\ell\widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)} + \frac{1}{4\rho^2}\|\bar\partial^\ell(\mathbf{h}^2)\|^2_{L^2(\mathbb{S}^1)}\Big]$$
$$\leqslant 2\rho^2\rho'\|\bar\partial^\ell\widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)} + C\rho'\|\mathbf{h}\|^2_{H^1(\mathbb{S}^1)}\|\bar\partial^\ell\mathbf{h}\|^2_{L^2(\mathbb{S}^1)}$$
$$\leqslant 2\rho^3\rho'\|\bar\partial^\ell\widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)} + C\sigma|||\mathbf{h}|||_T^2$$

which, in turn, implies that

$$\|\bar\partial^{\ell+1}\mathbf{h}\|^2_{L^2(\mathbb{S}^1)} + \frac{1}{2}\rho^2\rho'\|\bar\partial^\ell\mathbf{h}\|^2_{L^2(\mathbb{S}^1)} + \int_0^t \rho^3\|\bar\partial^\ell\mathbf{v}\|^2_{L^2(B_1)}ds - \int_0^t \rho^2\rho''\|\bar\partial^\ell\widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)}ds\mathrm{n}$$
$$\leqslant \|\bar\partial^\ell\mathbf{h}\|^2_{L^2(\mathbb{S}^1)} + C_{\delta,\delta_1}N(\|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)}, |||\mathbf{h}|||_T)$$
$$+ C_\delta\int_0^t \rho^3\|\mathbf{v}\|^2_{H^{\ell-1}(B_1)}ds + \delta\int_0^t \rho^3\|\mathbf{v}\|^2_{H^\ell(B_1)}ds + 2\delta_1|||\mathbf{h}|||_T^2 \quad (89)$$

if $C\sigma$ is chosen to be smaller than $2\delta_1$.

If $\ell = 1$, by (85) we find that

$$\|\bar\partial^2 \mathbf{h}\|^2_{L^2(\mathbb{S}^1)} + \frac{1}{2}\rho^2\rho'\|\bar\partial \mathbf{h}\|^2_{L^2(\mathbb{S}^1)} + \int_0^t \rho^3 \|\bar\partial \mathbf{v}\|^2_{L^2(B_1)} ds - \int_0^t \rho^2\rho''\|\bar\partial \widetilde{\mathbf{h}}\|^2_{L^2(\mathbb{S}^1)} ds \leqslant C_{\delta_1} N(\|\mathbf{h}_0\|_{H^\kappa(\mathbb{S}^1)}, |||\mathbf{h}|||_T) + \delta \int_0^t \rho^3 \|\mathbf{v}\|^2_{H^\ell(B_1)} ds + 2\delta_1 |||\mathbf{h}|||_T^2$$

which, combined with (62) and (85), implies that

$$\|\mathbf{h}\|^2_{H^2(\mathbb{S}^1)} + \frac{1}{2}\rho^2\rho'\|\mathbf{h}\|^2_{H^1(\mathbb{S}^1)} + c\int_0^t \rho^3 \|\bar\partial \mathbf{v}\|^2_{H^1(B_1)} ds - \int_0^t \rho^2\rho''\|\widetilde{\mathbf{h}}\|^2_{H^1(\mathbb{S}^1)} ds \leqslant C_{\delta_1} N(\|\mathbf{h}_0\|_{H^\kappa(\mathbb{S}^1)}, |||\mathbf{h}|||_T) + 2\delta_1 |||\mathbf{h}|||_T^2$$

for some constant $0 < c < 1$. A similar argument shows that

$$\|\mathbf{h}\|^2_{H^\kappa(\mathbb{S}^1)} + \frac{1}{2}\rho^2\rho'\|\mathbf{h}\|^2_{H^{\kappa-1}(\mathbb{S}^1)} + c\int_0^t \rho^3 \|\mathbf{v}\|^2_{H^{\kappa-1}(B_1)} ds - \int_0^t \rho^2\rho''\|\widetilde{\mathbf{h}}\|^2_{H^{\kappa-1}(\mathbb{S}^1)} ds \leqslant C_{\delta_1} N(\|\mathbf{h}_0\|_{H^\kappa(\mathbb{S}^1)}, |||\mathbf{h}|||_T) + C\delta_1 |||\mathbf{h}|||_T^2.$$

Finally, we look for an upper bound of $\int_0^t \rho^2\rho''\|\widetilde{\mathbf{h}}\|^2_{H^{\kappa-1}(\mathbb{S}^1)} ds$ to close the energy estimates. If (13) is satisfied, by interpolation and the inequality $\|\widetilde{\mathbf{h}}\|_{H^6(\mathbb{S}^1)} \leqslant C\|\mathbf{h}\|_{H^6(\mathbb{S}^1)}$,

$$\Big|\int_0^t \rho(s)^2\rho'(s)\|\widetilde{\mathbf{h}}(s)\|^2_{H^5(\mathbb{S}^1)} ds\Big| \leqslant C\int_0^\infty \frac{\rho(s)^3}{1+s}\|\widetilde{\mathbf{h}}(s)\|^{4/7}_{H^{2.5}(\mathbb{S}^1)}\|\widetilde{\mathbf{h}}(s)\|^{10/7}_{H^6(\mathbb{S}^1)} ds \leqslant C\Big[\int_0^\infty \frac{\rho(s)^{13/7}}{1+s} e^{-4\beta d(s)/7} ds\Big] N(\|\mathbf{h}_0\|_{H^\kappa(\mathbb{S}^1)}, |||\mathbf{h}|||_T) \leqslant C N(\|\mathbf{h}_0\|_{H^\kappa(\mathbb{S}^1)}, |||\mathbf{h}|||_T).$$

Now suppose that (15) is satisfied. If $\rho'' \leqslant 0$, then

$$\|\mathbf{h}(t)\|^2_{H^\kappa(\mathbb{S}^1)} + \frac{1}{2}\rho(t)^2\rho'(t)\|\widetilde{\mathbf{h}}(t)\|^2_{H^{\kappa-1}(\mathbb{S}^1)} + \int_0^t \rho(s)^3 \|\mathbf{v}(s)\|^2_{H^{\kappa-1}(B_1)} ds \leqslant C_{\delta_1} N(\|\mathbf{h}_0\|_{H^\kappa(\mathbb{S}^1)}, |||\mathbf{h}|||_T) + C\delta_1 |||\mathbf{h}|||_T^2. \tag{90}$$

If $\log\rho'$ has small total variation, then for some $\delta_2 \ll 1$,

$$\|(\log\rho')'\|_{L^1(0,\infty)} = \int_0^\infty \Big|\frac{\rho''(s)}{\rho'(s)}\Big| ds \leqslant \delta_2.$$

Since $\rho\sqrt{\rho'}\|\widetilde{\mathbf{h}}\|_{H^{\kappa-1}(\mathbb{S}^1)} \leqslant |||\mathbf{h}|||_T$,

$$\Big|\int_0^t \rho(s)^2\rho'(s)\|\widetilde{\mathbf{h}}(s)\|^2_{H^{\kappa-1}(\mathbb{S}^1)} ds\Big| \leqslant \Big[\int_0^\infty \frac{|\rho''(s)|}{\rho'(s)} ds\Big] |||\mathbf{h}|||_T^2 \leqslant \delta_2 |||\mathbf{h}|||_T^2;$$

thus we obtain (90) again with $\delta_1$ replaced by $\delta_1 + \delta_2$.

REMARK 6.1 By the energy estimate (90), we see that the quantity $\|\widetilde{\mathbf{h}}\|_{H^{\kappa-1}(\mathbb{S}^1)}$ decays at the rate $\frac{1}{\rho\sqrt{\rho'}}$. This decay rate is slower than the decay of $\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}$ if $\rho$ grows algebraically.

### 6.2 *Total norm bounds*

Combining (79), (80) and (90), by choosing $\delta_1 > 0$ small enough we conclude that

$$|||h|||_T^2 \leqslant C\,N(\|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)}, |||\mathbf{h}|||_T)$$

or

$$|||\mathbf{h}|||_T \leqslant C\Big[\|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)} + |||\mathbf{h}|||_T^2 + |||\mathbf{h}|||_T^{2p}\Big] \tag{91}$$

for some $p \in \mathbb{N}$, provided that condition (13) or (15), as well as the assumption (39), are valid. The constant $\sigma$ is chosen sufficiently small so as to absorb certain error terms on the right-hand side of our energy estimates by the left-hand side energy terms. In other words, as long as all of the constants we use in our elliptic estimates and the Sobolev embedding theorem are fixed, the maximum of $\sigma$ is a fixed computable number which can be chosen independent of $\|\mathbf{h}_0\|_{H^6(\mathbb{S}^1)}$. Inequality (91) then implies that there exists $\epsilon > 0$ small enough such that if $\|\mathbf{h}_0\|_{H^{\mathrm{K}}(\mathbb{S}^1)} \leqslant \epsilon$, by the continuity (in time) of $\mathbf{h}$,

$$|||\mathbf{h}|||_T \ll \sigma. \tag{92}$$

This suggests that as long as the solution exists, $\mathbf{h}$ has to satisfy the estimate (92), and this establishes Theorem 1.1 and 1.2.

## 7. The case with zero surface tension

### 7.1 *The ALE formulation*

Let $\widetilde{\mathbf{h}} : \Gamma(t) \to \mathbb{R}$ denote the signed height function defined by

$$\widetilde{\mathbf{h}}(\mathbf{x}, t) = \mathbf{h}\Big(\frac{\mathbf{x}}{|\mathbf{x}|}, t\Big) \quad \forall\, \mathbf{x} \in \Gamma(t).$$

Define $\bar{u} = -\nabla \bar{p} = \dfrac{\mathbf{x}}{|\mathbf{x}|^2 \log \rho}$ (which is the solution for $\Omega = \mathbb{A}$ in (5f)), as well as $\mathbf{u} = u - \bar{u}$ and $\mathbf{p} = p - \bar{p}$ . Then $(\mathbf{u}, \mathbf{p})$ satisfies

$$\begin{aligned}
\mathbf{u} + \nabla \mathbf{p} &= 0 && \text{in} \quad \Omega(t), && \text{(93a)}\\
\operatorname{div}\mathbf{u} &= 0 && \text{in} \quad \Omega(t), && \text{(93b)}\\
\mathbf{p} &= 0 && \text{on} \quad \mathbb{S}^1, && \text{(93c)}\\
\mathbf{p} &= \frac{\log(1 + \rho^{-1}\widetilde{\mathbf{h}})}{\log \rho} && \text{on} \quad \Gamma(t). && \text{(93d)}
\end{aligned}$$

Now we rewrite (93) using the ALE coordinate. The ALE map $\psi : \mathbb{A}_{1/\rho} \to \Omega(t)$ is taken to be the solution to

$$\begin{aligned}
\Delta \psi &= 0 && \text{in} \quad \mathbb{A}_{1/\rho}, && \text{(94a)}\\
\psi &= \rho e && \text{on} \quad \partial B_{1/\rho}, && \text{(94b)}\\
\psi &= (\rho + \mathbf{h})e && \text{on} \quad \mathbb{S}^1 \times (0, T), && \text{(94c)}
\end{aligned}$$

where we recall that $e$ denotes the identity map given by $e(x) = x$, and $\mathbb{A}_{1/\rho}$ is the annular region enclosed by $\partial B_{1/\rho}$ and $\mathbb{S}^1$. We also remark that unlike the case considered in the previous sections, the domain of the ALE map now is time-dependent, but a priori known, because of (8). Following the analysis of the Euler equations in [6], we define $\mathbf{v} = \mathbf{u} \circ \psi$, $\mathbf{q} = \mathbf{p} \circ \psi$, and $\mathrm{A} = (\nabla\psi)^{-1}$. Then $(\mathbf{v}, \mathbf{q})$ satisfies

$$\mathbf{v}^i + \mathrm{A}_i^j \mathbf{q},_j = 0 \quad \text{in} \quad \mathbb{A}_{1/\rho}, \tag{95a}$$

$$\mathrm{A}_i^j \mathbf{v}^i,_j = 0 \quad \text{in} \quad \mathbb{A}_{1/\rho}, \tag{95b}$$

$$\mathbf{q} = 0 \quad \text{on} \quad \partial B_{1/\rho}, \tag{95c}$$

$$\mathbf{q} = \frac{\log(1+\rho^{-1}\mathbf{h})}{\log\rho} \quad \text{on} \quad \mathbb{S}^1 \times (0,T). \tag{95d}$$

Since $\log(1+x) = x + \mathcal{O}(x^2)$ if $x \ll 1$, (95d) implies that

$$\mathbf{q} = \frac{\mathbf{h}}{\rho\log\rho} + \frac{\delta\mathbf{q}}{\rho^2\log\rho} \quad \text{on} \quad \mathbb{S}^1 \tag{95d'}$$

with $\delta\mathbf{q} = \rho^2\big[\log(1+\rho^{-1}\mathbf{h}) - \rho^{-1}\mathbf{h}\big]$ satisfying for $k \geqslant 0$,

$$\|\delta\mathbf{q}\|_{H^k(\mathbb{S}^1)} \leqslant C\|\mathbf{h}\|_{H^1(\mathbb{S}^1)}\|\mathbf{h}\|_{H^k(\mathbb{S}^1)}. \tag{96}$$

7.1.1 *The evolution equation of* **h**. Using (24), (29) and (32), we find that

$$\mathbf{h}_t + \rho' = (u\circ\psi)\cdot\Big[\mathrm{N} - \frac{\mathbf{h}_\theta}{\rho+\mathbf{h}}\mathrm{T}\Big] = \mathbf{v}\cdot\Big[\mathrm{N} - \frac{\mathbf{h}_\theta}{\rho+\mathbf{h}}\mathrm{T}\Big] + \frac{1}{(\rho+\mathbf{h})\log\rho}. \tag{97}$$

By (7),

$$\mathbf{h}_t = \frac{\mathrm{J}\mathrm{A}_i^j\mathbf{v}^i\mathrm{N}_j}{\rho+\mathbf{h}} + \frac{1}{(\rho+\mathbf{h})\log\rho} - \frac{1}{\rho\log\rho} = \frac{\mathrm{J}\mathrm{A}_i^j\mathbf{v}^i\mathrm{N}_j}{\rho+\mathbf{h}} - \frac{\mathbf{h}}{\rho(\rho+\mathbf{h})\log\rho}$$

which further implies that

$$(\rho+\mathbf{h})\mathbf{h}_t = \mathrm{J}\mathrm{A}_i^j\mathbf{v}^i\mathrm{N}_j - \frac{\mathbf{h}}{\rho\log\rho} = \mathrm{J}\mathrm{A}_i^j\mathbf{v}^i\mathrm{N}_j - \rho'\mathbf{h}$$

or equivalently,

$$(\rho\mathbf{h})_t = \mathrm{J}\mathrm{A}_i^j\mathbf{v}^i\mathrm{N}_j + R, \tag{98}$$

where

$$R = -\mathbf{h}\mathbf{h}_t = -\frac{\mathbf{h}}{\rho+\mathbf{h}}\mathrm{J}\mathrm{A}_i^j\mathbf{v}^i\mathrm{N}_j + \frac{\mathbf{h}^2}{\rho(\rho+\mathbf{h})\log\rho}.$$

7.2 *The total norm and the bootstrap assumption*

We now define the total norm $|||\cdot|||_T$, used to establish (12), as follows:

$$|||\mathbf{h}|||_T \equiv \Big[ \int_0^T \frac{1}{\log \rho(t)} \|\mathbf{h}(t)\|^2_{H^{3.5}(\mathbb{S}^1)} dt \Big]^{\frac{1}{2}} + \sup_{t\in[0,T]} \rho(t)\|\mathbf{h}(t)\|_{H^3(\mathbb{S}^1)}. \tag{99}$$

We note that the definition of total norm implies that

$$\|\mathbf{h}\|_{H^3(\mathbb{S}^1)} \leqslant \frac{1}{\rho} |||\mathbf{h}|||_T. \tag{100}$$

Our estimates are founded upon the basic bootstrapping assumption that

$$\|\mathbf{h}\|_{H^3(\mathbb{S}^1)} \leqslant \sigma \ll 1. \tag{101}$$

7.3 *Preliminaries*

7.3.1 *Estimates of $D\psi$,* A *and* J. Similar to the estimates established in Section 4.1, we have the same estimates for $\psi$, A and J with the domain $\mathbb{A}_{1/\rho}$ replacing the domain $B_1$. We refer the reader to Section 4.1 for those estimates.

7.3.2 *Estimates of* **v** *and* **q** *in terms of* **h**.

LEMMA 7.1 Let $(\mathbf{v}, \mathbf{q})$ be the solution to (95). Then for all $k \geqslant 0$,

$$\|\mathbf{v}\|_{H^k(\mathbb{A}_{1/\rho})} + \rho^{-1}\|\mathbf{q}\|_{H^{k+1}(\mathbb{A}_{1/\rho})} \leqslant \frac{C}{\rho^2 \log \rho} \|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)} \tag{102}$$

for some constant $C > 0$.

*Proof.* Note that **q** satisfies

$$\begin{aligned} \mathrm{A}_i^k(\mathrm{A}_i^j \mathbf{q},_j),_k &= 0 && \text{in} \quad \mathbb{A}_{1/\rho}, \\ \mathbf{q} &= 0 && \text{on} \quad \partial B_{1/\rho}, \\ \mathbf{q} &= \frac{\mathbf{h}}{\rho \log \rho} + \frac{\delta \mathbf{q}}{\rho^2 \log \rho} && \text{on} \quad \mathbb{S}^1 \times (0, T), \end{aligned}$$

or equivalently,

$$\Delta \mathbf{q} = (\delta_i^k \delta_i^j - \rho^2 \mathrm{A}_i^k \mathrm{A}_i^j)\mathbf{q},_{jk} + \rho^2 \mathrm{A}_i^k \mathrm{A}_r^j \psi^r,_{ks} \mathrm{A}_i^s \mathbf{q},_j \quad \text{in} \quad \mathbb{A}_{1/\rho}, \tag{103a}$$

$$\mathbf{q} = 0 \quad \text{on} \quad \partial B_{1/\rho}, \tag{103b}$$

$$\mathbf{q} = \frac{\mathbf{h}}{\rho \log \rho} + \frac{\delta \mathbf{q}}{\rho^2 \log \rho} \quad \text{on} \quad \mathbb{S}^1 \times (0, T). \tag{103c}$$

Elliptic estimates together with (41), (43), (42) and (96) then imply that

$$\|\mathbf{q}\|_{H^2(\mathbb{A}_{1/\rho})} \leqslant C\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{q}\|_{H^2(\mathbb{A}_{1/\rho})} + \frac{C}{\rho \log \rho}\big(1 + \|\mathbf{h}\|_{H^1(\mathbb{S}^1)}\big)\|\mathbf{h}\|_{H^{1.5}(\mathbb{S}^1)};$$

thus the bootstrapping assumption (101) shows that

$$\|\mathbf{q}\|_{H^2(\mathbb{A}_{1/\rho})} \leqslant \frac{C}{\rho\log\rho}\|\mathbf{h}\|_{H^{1.5}(\mathbb{S}^1)}. \tag{104}$$

Similarly, elliptic estimates imply that

$$\begin{aligned}\|\mathbf{q}\|_{H^{k+1}(\mathbb{A}_{1/\rho})} \leqslant C\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{q}\|_{H^{k+1}(\mathbb{A}_{1/\rho})} + C\rho^2\sum_{\ell=1}^{s-1}\left\|D^\ell(\mathrm{AA})D^{s-\ell+1}\mathbf{q}\right\|_{L^2(\mathbb{A}_{1/\rho})}\\ + C\rho^2\left\|\mathrm{AAA}D^2\psi D\mathbf{q}\right\|_{H^{k-1}(\mathbb{A}_{1/\rho})} + \frac{C}{\rho\log\rho}\big(1+\|\mathbf{h}\|_{H^1(\mathbb{S}^1)}\big)\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)};\end{aligned}$$

thus

$$\begin{aligned}\|\mathbf{q}\|_{H^{k+1}(\mathbb{A}_{1/\rho})} \leqslant C\rho^2\sum_{\ell=1}^{s-1}\left\|D^\ell(\mathrm{AA})D^{s-\ell+1}\mathbf{q}\right\|_{L^2(\mathbb{A}_{1/\rho})} + C\rho^2\left\|\mathrm{AAA}D^2\psi D\mathbf{q}\right\|_{H^{k-1}(\mathbb{A}_{1/\rho})}\\ + \frac{C}{\rho\log\rho}\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)}.\end{aligned}$$

By induction we conclude that

$$\|\mathbf{q}\|_{H^{k+1}(\mathbb{A}_{1/\rho})} \leqslant \frac{C}{\rho\log\rho}\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)}.$$

As a consequence, (95a) further implies that

$$\begin{aligned}\|\mathbf{v}\|_{H^k(\mathbb{A}_{1/\rho})} &\leqslant C\Big[\|D\mathrm{A}\|_{H^{k-1}(\mathbb{A}_{1/\rho})}\|D\mathbf{q}\|_{L^\infty(\mathbb{A}_{1/\rho})} + \|\mathrm{A}\|_{L^\infty(\mathbb{A}_{1/\rho})}\|D\mathbf{q}\|_{H^k(\mathbb{A}_{1/\rho})}\Big]\\ &\leqslant \frac{C}{\rho^2\log\rho}\|\mathbf{h}\|_{H^{k+0.5}(\mathbb{S}^1)}\end{aligned} \tag{105}$$

which concludes proof of the lemma. □

7.3.3 *Estimates of* **h** *in terms of* **v**. Having established (105), we next proof the reverse inequality for the case $2 \leqslant k \leqslant 3$.

LEMMA 7.2 For $2 \leqslant k \leqslant 3$,

$$\frac{1}{\rho^2\log\rho}\|\mathbf{h}\|_{H^{k+0.5}\mathbb{S}^1)} \leqslant C\|\mathbf{v}\|_{H^k(\mathbb{A}_{1/\rho})} \tag{106}$$

for some constant $C > 0$.

*Proof.* Since $u = \mathbf{u} + \bar{u}$ and $\mathbf{v} = \mathbf{u}\circ\psi$, (5a) implies that

$$\mathbf{v} + \bar{u}\circ\psi + \mathrm{A}^{\mathrm{T}}\nabla(p\circ\psi) = 0 \qquad \text{in} \quad \mathbb{A}_{1/\rho}. \tag{107}$$

By the chain rule, $\dfrac{\partial}{\partial x_j} = \mathrm{N}_j \dfrac{\partial}{\partial r} + \dfrac{\mathrm{T}_j}{|\mathbf{x}|}\dfrac{\partial}{\partial \theta}$; thus

$$\mathrm{A}^{\mathrm{T}}\nabla(p\circ\psi)\cdot\frac{\partial\psi}{\partial\theta} = \mathrm{A}_i^j (p\circ\psi),_j \frac{\partial\psi^i}{\partial\theta} = \mathrm{A}_i^j\frac{\partial\psi^i}{\partial\theta}\Big[\mathrm{N}_j\frac{\partial(p\circ\psi)}{\partial r} + \frac{\mathrm{T}_j}{|\mathbf{x}|}\frac{\partial(p\circ\psi)}{\partial\theta}\Big].$$

Therefore, due to the fact that $\mathrm{JA}^{\mathrm{T}}\mathrm{N}\perp\psi_\theta$ on $\mathbb{S}^1$, we conclude from (5c) that

$$\mathrm{A}^{\mathrm{T}}\nabla(p\circ\psi)\cdot\frac{\partial\psi}{\partial\theta} = 0 \qquad \text{on} \quad \mathbb{S}^1.$$

which, by taking the inner product of (107) with $\dfrac{\partial\psi}{\partial\theta}$, shows that

$$\Big[\mathbf{v} + \frac{\mathrm{N}}{(\rho+\mathbf{h})\log\rho}\Big]\cdot\frac{\partial\psi}{\partial\theta} = 0 \qquad \text{on} \quad \mathbb{S}^1\times(0,T).$$

As a consequence,

$$\Big[\mathbf{v} + \frac{\mathrm{N}}{(\rho+\mathbf{h})\log\rho}\Big]\cdot\frac{\partial^2\psi}{\partial\theta^2} = -\frac{\partial}{\partial\theta}\Big[\mathbf{v} + \frac{\mathrm{N}}{(\rho+\mathbf{h})\log\rho}\Big]\cdot\frac{\partial\psi}{\partial\theta} \qquad \text{on} \quad \mathbb{S}^1. \tag{108}$$

Since $\dfrac{\partial\psi}{\partial\theta} = \mathbf{h}_\theta\mathrm{N} + (\rho+\mathbf{h})\mathrm{T}$, letting $\mathrm{E}\equiv\mathbf{v}\cdot\mathrm{N} - \dfrac{\mathbf{h}}{\rho(\rho+\mathbf{h})\log\rho}$, (108) implies that

$$\frac{1}{\rho\log\rho}\mathbf{h}_{\theta\theta} = \mathrm{E}\mathbf{h}_{\theta\theta} + \mathbf{v}\cdot\big[(\rho+\mathbf{h})\mathrm{N} - 2\mathbf{h}_\theta\mathrm{T}\big] - \mathbf{v}_\theta\cdot\psi_\theta + \frac{1}{\log\rho}\frac{\mathbf{h}_\theta^2}{(\rho+\mathbf{h})^2} \quad \text{on} \quad \mathbb{S}^1.$$

We note that by (102), for $0.5\leqslant k\leqslant 1.5$, E satisfies

$$\|\mathrm{E}\|_{H^k(\mathbb{S}^1)} \leqslant C\Big[\|\mathbf{v}\|_{H^{k+0.5}(\mathbb{A}_{1/\rho})} + \frac{1}{\rho^2\log\rho}\|\mathbf{h}\|_{H^k(\mathbb{S}^1)}\Big] \leqslant \frac{C}{\rho^2\log\rho}\|\mathbf{h}\|_{H^{k+1}(\mathbb{S}^1)}.$$

By elliptic estimates and (102), we obtain that

$$\begin{aligned}\frac{1}{\rho\log\rho}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)} &\leqslant C\Big[\big(\|\mathrm{E}\|_{H^{1.5}(\mathbb{S}^1)}\|\mathbf{h}\|_{H^3(\mathbb{S}^1)} + \|\mathrm{E}\|_{H^1(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)}\big)\\ &\qquad + \rho\|\mathbf{v}\|_{H^3(\mathbb{A}_{1/\rho})} + \|\mathbf{v}\|_{H^{2.5}(\mathbb{A}_{1/\rho})}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} + \frac{1}{\rho^2\log\rho}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}\Big]\\ &\leqslant C\Big[\frac{1}{\rho^2\log\rho}\big(\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}\|\mathbf{h}\|_{H^3(\mathbb{S}^1)} + \|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)}\big) + \rho\|\mathbf{v}\|_{H^3(\mathbb{A}_{1/\rho})}\Big];\end{aligned}$$

thus, using the bootstrap assumption (101), we have that

$$\frac{1}{\rho\log\rho}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)} \leqslant C\rho\|\mathbf{v}\|_{H^3(\mathbb{A}_{1/\rho})}.$$

Similarly,

$$\begin{aligned}\frac{1}{\rho\log\rho}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} &\leqslant C\Big[\|\mathrm{E}\|_{H^1(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} + \rho\|\mathbf{v}\|_{H^2(\mathbb{A}_{1/\rho})} + \frac{1}{\rho^2\log\rho}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{1.5}(\mathbb{S}^1)}\Big]\\ &\leqslant C\Big[\frac{1}{\rho^2\log\rho}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} + \rho\|\mathbf{v}\|_{H^2(\mathbb{A}_{1/\rho})}\Big]\end{aligned}$$

from which we conclude that

$$\frac{1}{\rho\log\rho}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} \leqslant C\rho\|\mathbf{v}\|_{H^2(\mathbb{A}_{1/\rho})}.$$

Estimate (106) then follows from linear interpolation. □

7.3.4 *The estimate of $R$.* Recall the evolution equation of $\mathbf{h}$ is given by

$$(\rho\mathbf{h})_t = \mathrm{JA}_i^j\,\mathbf{v}^i\mathrm{N}_j + R$$

with $R = -\frac{\mathbf{h}}{\rho+\mathbf{h}}\mathrm{JA}_i^j\,\mathbf{v}^i\mathrm{N}_j + \frac{\mathbf{h}^2}{\rho(\rho+\mathbf{h})\log\rho}$. By the normal trace theorem,

$$\|\mathrm{JA}_i^j\,\mathbf{v}^i\mathrm{N}_j\|_{H^{-0.5}(\mathbb{S}^1)} \leqslant C\Big[\|\mathrm{JA}\mathbf{v}\|_{L^2(\mathbb{A}_{1/\rho})} + \|\mathrm{JA}_i^j\,\mathbf{v}^i,_j\|_{L^2(\mathbb{A}_{1/\rho})}\Big] \leqslant C\rho\|\mathbf{v}\|_{L^2(\mathbb{A}_{1/\rho})}; \tag{109}$$

thus (102) implies that

$$\begin{aligned}\|R\|_{H^{-0.5}(\mathbb{S}^1)} &\leqslant C\|\mathbf{h}\|_{H^1(\mathbb{S}^1)}\|\mathbf{v}\|_{L^2(\mathbb{A}_{1/\rho})} + \frac{C}{\rho^2\log\rho}\|\mathbf{h}\|^2_{L^4(\mathbb{S}^1)}\\ &\leqslant \frac{C}{\rho^2\log\rho}\|\mathbf{h}\|_{H^1(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{0.5}(\mathbb{S}^1)}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|R\|_{H^{2.5}(\mathbb{S}^1)} &\leqslant C\rho^{-1}\Big[\|\mathbf{h}\|_{H^1(\mathbb{S}^1)}\|\mathrm{JA}\mathbf{v}\|_{H^3(\mathbb{A}_{1/\rho})} + \|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)}\|\mathrm{JA}\mathbf{v}\|_{H^{1.5}(\mathbb{A}_{1/\rho})}\Big]\\ &\quad + \frac{C}{\rho^2\log\rho}\|\mathbf{h}\|_{H^1(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{2.5}(\mathbb{S}^1)} \leqslant \frac{C}{\rho^2\log\rho}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{3.5}(\mathbb{S}^1)}.\end{aligned}$$

It then follows from interpolation that

$$\|R\|_{H^k(\mathbb{S}^1)} \leqslant \frac{C}{\rho^2\log\rho}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{k+1}(\mathbb{S}^1)}. \tag{110}$$

### 7.4 *Energy estimates in the space of higher regularity*

7.4.1 *The $L^2$-estimate.* Taking the $L^2$ inner-produce of equation (95a) with $\mathbf{J}\mathbf{v}$, we find that

$$\int_{\mathbb{A}_{1/\rho}} \mathrm{J}|\mathbf{v}|^2 dx + \int_{\mathbb{A}_{1/\rho}} \mathrm{JA}_i^j\,\mathbf{q},_j\,\mathbf{v}^i\,dx = 0.$$

Integrating by parts in $x_j$, the boundary conditions (95c) and (95d') show that

$$\int_{\mathbb{A}_{1/\rho}} \mathrm{J}|\mathbf{v}|^2 dx + \int_{\mathbb{S}^1} \frac{\mathbf{h}}{\rho\log\rho}\mathrm{JA}_i^j\,\mathbf{v}^i\mathrm{N}_j\,dS = -\frac{1}{\rho^2\log\rho}\int_{\mathbb{S}^1}\mathrm{JA}_i^j\,\delta\mathbf{q}\mathbf{v}^i\mathrm{N}_j\,dS.$$

By the evolution equation (98) and $H^{0.5}(\mathbb{S}^1)$-$H^{-0.5}(\mathbb{S}^1)$ duality,

$$\begin{aligned}&\int_{\mathbb{A}_{1/\rho}} \mathrm{J}|\mathbf{v}|^2 dx + \int_{\mathbb{S}^1}\frac{\mathbf{h}}{\rho\log\rho}(\rho\mathbf{h})_t\,dS\\ &\qquad\leqslant \frac{C}{\rho\log\rho}\|\mathbf{h}\|_{H^{0.5}(\mathbb{S}^1)}\|R\|_{H^{-0.5}(\mathbb{S}^1)} + \frac{C}{\rho^2\log\rho}\|\delta\mathbf{q}\|_{H^1(\mathbb{S}^1)}\|\mathrm{JA}_i^j\,\mathbf{v}^i\mathrm{N}_j\|_{H^{-0.5}(\mathbb{S}^1)}.\end{aligned}$$

From (45), $\mathrm{J} \geqslant \dfrac{\rho^2}{2}$; hence (96), (109) and (110) imply that

$$\frac{\rho^2}{2}\|\mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})} + \frac{1}{2\rho^2 \log\rho}\frac{d}{dt}\|\rho\mathbf{h}\|^2_{L^2(\mathbb{S}^1)} \leqslant \frac{C}{\rho^2(\log\rho)^2}\|\mathbf{h}\|^2_{H^1(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{0.5}(\mathbb{S}^1)}.$$

As a consequence, by (100) we obtain that

$$\rho^4\log\rho\|\mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})} + \frac{d}{dt}\|\rho\mathbf{h}\|^2_{L^2(\mathbb{S}^1)} \leqslant \frac{C}{\rho^3\log\rho}|\!|\!|\mathbf{h}|\!|\!|^3_T;$$

thus integrating by parts in time over $(0,t)$ give us

$$\int_0^t \rho^4\log\rho\|\mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})}ds + \rho(t)^2\|\mathbf{h}(t)\|^2_{L^2(\mathbb{S}^1)} \leqslant r_0^2\|\mathbf{h}_0\|^2_{L^2(\mathbb{S}^1)} + C|\!|\!|\mathbf{h}|\!|\!|^3_T, \tag{111}$$

where we have used (9) and (10) to conclude the uniform boundedness of the integral $\int_0^t \frac{1}{\rho^3\log\rho}ds$ uniformly in $t>0$.

7.4.2 *The estimate in the higher-order energy space.* Let $\bar\partial = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ denote the tangential derivative. Tangentially differentiating (95a) $\ell$-times ($\ell = 1,2,3$) and then computing the $L^2$ inner-product of the resulting equation with $\mathrm{J}\bar\partial^\ell\mathbf{v}$, we find that

$$\int_{\mathbb{A}_{1/\rho}} \mathrm{J}|\bar\partial^\ell\mathbf{v}|^2dx + \int_{\mathbb{A}_{1/\rho}} \mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell\mathbf{q},_j\,\bar\partial^\ell\mathbf{v}^i\,dx = -\sum_{k=1}^{\ell}\binom{\ell}{k}\int_{\mathbb{A}_{1/\rho}} \mathrm{J}(\bar\partial^k\mathrm{A}_i^j)(\bar\partial^{\ell-k}\mathbf{q},_j)\bar\partial^\ell\mathbf{v}^i\,dx.$$

Since $\mathbf{q} = -(\nabla\psi)^{\mathrm{T}}\mathbf{v}$, obtained by multiplying (95a) by $(\nabla\psi)^{\mathrm{T}}$,

$$\begin{aligned}
\Big|\int_{\mathbb{A}_{1/\rho}} \mathrm{J}\mathrm{A}_i^j\Big[\big[\bar\partial^\ell\mathbf{q},_j - (\bar\partial^\ell\mathbf{q}),_j\big]\bar\partial^\ell\mathbf{v}^i + \bar\partial^\ell\mathbf{q}\big[(\bar\partial^\ell\mathbf{v}^i),_j - \bar\partial^\ell\mathbf{v}^i,_j\big]\Big]dx\Big|&\\
\leqslant C\rho\|\mathbf{q}\|_{H^\ell(\mathbb{A}_{1/\rho})}\|\mathbf{v}\|_{H^\ell(\mathbb{A}_{1/\rho})}&\\
\leqslant C\rho\Big[\rho\|\mathbf{v}\|_{H^{\ell-1}(\mathbb{A}_{1/\rho})} + \|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)}\|\mathbf{v}\|_{L^\infty(\mathbb{A}_{1/\rho})}\Big]\|\mathbf{v}\|_{H^\ell(\mathbb{A}_{1/\rho})}.&
\end{aligned}$$

Hence, the Hodge elliptic estimate (62), (102) and Young's inequality imply that

$$\begin{aligned}
\int_{\mathbb{A}_{1/\rho}} \mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell\mathbf{q},_j\,\bar\partial^\ell\mathbf{v}^i\,dx &\geqslant \int_{\mathbb{S}^1}\mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell\mathbf{q}\bar\partial^\ell\mathbf{v}^i\mathrm{N}_j\,dS - \int_{\mathbb{A}_{1/\rho}}\mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell\mathbf{q}\bar\partial^\ell\mathbf{v}^i,_j\,dx\\
&\quad - C_\delta\rho^2\|\mathbf{v}\|^2_{H^{\ell-1}(\mathbb{A}_{1/\rho})} - C_\delta\|\mathbf{h}\|^2_{H^{\ell+0.5}(\mathbb{S}^1)}\|\mathbf{v}\|^2_{H^2(\mathbb{A}_{1/\rho})} - \delta\rho^2\|\mathbf{v}\|^2_{H^\ell(\mathbb{A}_{1/\rho})}\\
&\geqslant \int_{\mathbb{S}^1}\mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell\Big[\frac{\mathbf{h}}{\rho\log\rho} + \frac{\delta\mathbf{q}}{\rho^2\log\rho}\Big]\bar\partial^\ell\mathbf{v}^i\mathrm{N}_j\,dS - \int_{\mathbb{A}_{1/\rho}}\mathrm{J}\mathrm{A}_i^j\,\bar\partial^\ell\mathbf{q}\bar\partial^\ell\mathbf{v}^i,_j\,dx\\
&\quad - C_\delta\rho^2\sum_{k=0}^{\ell-1}\|\bar\partial^k\mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})} - \frac{C_\delta}{\rho^4(\log\rho)^2}\|\mathbf{h}\|^2_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|^2_{H^{\ell+0.5}(\mathbb{S}^1)} - \delta\rho^2\|\bar\partial^\ell\mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})}.
\end{aligned}$$

By the divergence-free condition (95b), $A^j_i\bar\partial^\ell \mathbf{v}^i,_j = -\sum_{k=1}^\ell \binom{\ell}{k}\bar\partial^k A^j_i\bar\partial^{\ell-k}\mathbf{v}^i,_j$ . As a consequence, it follows from (102) that

$$\begin{aligned}\int_{\mathbb{A}_{1/\rho}} JA^j_i\bar\partial^\ell\mathbf{q}\bar\partial^\ell\mathbf{v}^i,_j\,dx &\leqslant C\rho^2\|\mathbf{q}\|_{H^{\ell+0.5}(\mathbb{A}_{1/\rho})}\times\\ &\quad\times\Big[\|DA\|_{H^{0.5}(\mathbb{A}_{1/\rho})}\|\mathbf{v}\|_{H^\ell(\mathbb{A}_{1/\rho})}+\|\mathbf{v}\|_{H^{1.5}(\mathbb{A}_{1/\rho})}\|DA\|_{H^{\ell-1}(\mathbb{A}_{1/\rho})}\Big]\\ &\leqslant \frac{C}{\rho\log\rho}\|\mathbf{h}\|_{H^\ell(\mathbb{S}^1)}\Big[\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{v}\|_{H^\ell(\mathbb{A}_{1/\rho})}+\|\mathbf{v}\|_{H^{1.5}(\mathbb{A}_{1/\rho})}\|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)}\Big]\\ &\leqslant \frac{C}{\rho^3(\log\rho)^2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^\ell(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)}.\end{aligned}$$

Once again using the normal trace theorem, we find that

$$\begin{aligned}\int_{\mathbb{S}^1} JA^j_i\bar\partial^\ell\delta\mathbf{q}\bar\partial^\ell\mathbf{v}^i N_j\,dS &\leqslant C\|\bar\partial^\ell\delta\mathbf{q}\|_{H^{0.5}(\mathbb{S}^1)}\|JA^j_i\bar\partial^\ell\mathbf{v}^iN_j\|_{H^{-0.5}(\mathbb{S}^1)}\\ &\leqslant C\|\mathbf{h}\|_{H^1(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)}\Big[\|JA^T\bar\partial^\ell\mathbf{v}\|_{L^2(\mathbb{A}^\rho)}+\|JA^j_i(\bar\partial^\ell\mathbf{v}^i),_j\|_{L^2(\mathbb{A}_{1/\rho})}\Big]\\ &\leqslant C\|\mathbf{h}\|_{H^1(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)}\Big[\rho\|\mathbf{v}\|_{H^\ell(\mathbb{A}_{1/\rho})}+\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{v}\|_{H^\ell(\mathbb{A}_{1/\rho})}\\ &\qquad\qquad+\|\mathbf{v}\|_{H^{1.5}(\mathbb{A}_{1/\rho})}\|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)}\Big]\\ &\leqslant \frac{C}{\rho^2\log\rho}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|^2_{H^{\ell+0.5}(\mathbb{S}^1)}.\end{aligned}$$

Moreover, by (44), (45) and (102),

$$\begin{aligned}-\sum_{k=1}^{\ell}\binom{\ell}{k}\int_{\mathbb{A}_{1/\rho}} J(\bar\partial^kA^j_i)(\bar\partial^{\ell-k}\mathbf{q},_j)\bar\partial^\ell\mathbf{v}^i\,dx &\leqslant C\|J\|_{L^\infty(\mathbb{A}_{1/\rho})}\times\\ \times\Big[\|D^\ell A\|_{L^2(\mathbb{A}_{1/\rho})}\|D\mathbf{q}\|_{L^\infty(\mathbb{A}_{1/\rho})}+\|DA\|_{L^4(\mathbb{A}_{1/\rho})}\|D^\ell\mathbf{q}\|_{L^4(\mathbb{A}_{1/\rho})}\Big]&\|\mathbf{v}\|_{H^\ell(\mathbb{A}_{1/\rho})}\\ &\leqslant \frac{C}{\rho^3(\log\rho)^2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|_{H^\ell(\mathbb{S}^1)}\|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)},\end{aligned}$$

so that

$$\begin{aligned}\int_{\mathbb{A}_{1/\rho}} J|\bar\partial^\ell\mathbf{v}|^2dx+\frac{1}{\rho\log\rho}\int_{\mathbb{S}^1}JA^j_i\bar\partial^\ell\mathbf{h}\bar\partial^\ell\mathbf{v}^iN_j\,dS &\leqslant \frac{C_\delta}{\rho^3(\log\rho)^2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|^2_{H^{\ell+0.5}(\mathbb{S}^1)}\\ &\quad+C_\delta\rho^2\sum_{k=0}^{\ell-1}\|\bar\partial^k\mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})}+\delta\rho^2\|\bar\partial^\ell\mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})}.\end{aligned}\tag{112}$$

By the evolution equation (98),

$$\begin{aligned}JA^j_i\bar\partial^\ell\mathbf{v}^iN_j &= \bar\partial^\ell\big[JA^j_i\mathbf{v}^iN_j\big]-\sum_{k=1}^{\ell}\binom{\ell}{k}\bar\partial^k(JA^TN)\cdot\bar\partial^{\ell-k}\mathbf{v}\\ &= \bar\partial^\ell(\rho\mathbf{h})_t+\bar\partial^\ell R-\sum_{k=1}^{\ell}\binom{\ell}{k}\bar\partial^k(JA^TN)\cdot\bar\partial^{\ell-k}\mathbf{v}.\end{aligned}$$

Therefore, by choosing $\delta > 0$ small enough, we obtain that

$$\begin{aligned}
&\frac{\rho^2}{2}\|\bar\partial^\ell \mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})} + \frac{1}{2\rho^2\log\rho}\frac{d}{dt}\|\rho\bar\partial^\ell \mathbf{h}\|^2_{L^2(\mathbb{S}^1)} \\
&\qquad \leqslant \frac{C}{\rho\log\rho}\|\mathbf{h}\|_{H^{\ell+0.5}(\mathbb{S}^1)}\|R\|_{H^{\ell-0.5}(\mathbb{S}^1)} + \frac{C}{\rho^3(\log\rho)^2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|^2_{H^{\ell+0.5}(\mathbb{S}^1)} \\
&\qquad\qquad + C\rho^2\sum_{k=0}^{\ell-1}\|\bar\partial^k \mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})} \\
&\qquad \leqslant \frac{C}{\rho^3(\log\rho)^2}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|^2_{H^{\ell+0.5}(\mathbb{S}^1)} + C\rho^2\sum_{k=0}^{\ell-1}\|\bar\partial^k \mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})}
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\rho^4\log\rho\|\bar\partial^\ell \mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})} + \frac{d}{dt}\|\rho\bar\partial^\ell h\|^2_{L^2(\mathbb{S}^1)} \\
&\qquad \leqslant \frac{C}{\rho\log\rho}\|\mathbf{h}\|_{H^2(\mathbb{S}^1)}\|\mathbf{h}\|^2_{H^{\ell+0.5}(\mathbb{S}^1)} + C\rho^4\log\rho\sum_{k=0}^{\ell-1}\|\bar\partial^k \mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})}. \qquad (113)
\end{aligned}$$

Let $\ell = 1$ in (113). By (100), for $t \in (0, T)$,

$$\begin{aligned}
&\int_0^t \rho^4\log\rho\|\bar\partial \mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})}ds + \rho(t)^2\|\bar\partial h(t)\|^2_{L^2(\mathbb{S}^1)} \\
&\qquad \leqslant r_0^2\|\mathbf{h}_0\|^2_{H^\ell(\mathbb{S}^1)} + C\int_0^t\Big[\frac{1}{\rho^4\log\rho}\|\!|\mathbf{h}|\!\|^3_T + \rho^4\log\rho\|\mathbf{v}\|^2_{L^2(\mathbb{A}_{1/\rho})}\Big]ds \\
&\qquad \leqslant r_0^2\|\mathbf{h}_0\|^2_{H^1(\mathbb{S}^1)} + C\|\!|\mathbf{h}|\!\|^3_T
\end{aligned}$$

which, combined with (62) and (111), further shows that

$$\int_0^t \rho^4\log\rho\|\mathbf{v}\|^2_{H^1(\mathbb{A}_{1/\rho})}ds + \rho(t)^2\|\mathbf{h}(t)\|^2_{H^1(\mathbb{S}^1)} \leqslant Cr_0^2\|\mathbf{h}_0\|^2_{H^1(\mathbb{S}^1)} + C\|\!|\mathbf{h}|\!\|^3_T. \qquad (114)$$

Similarly, letting $\ell = 2$ and $\ell = 3$ in (113), by (62) we find that

$$\begin{aligned}
&\int_0^t \rho^4\log\rho\|\mathbf{v}\|^2_{H^3(\mathbb{A}_{1/\rho})}ds + \rho(t)^2\|\mathbf{h}(t)\|^2_{H^4(\mathbb{S}^1)} \\
&\qquad \leqslant Cr_0^2\|\mathbf{h}_0\|^2_{H^3(\mathbb{S}^1)} + C\int_0^t\Big[\frac{1}{\rho^2\log\rho}\|\mathbf{h}\|^2_{H^{3.5}(\mathbb{S}^1)} + \rho^4\log\rho\|\mathbf{v}\|^2_{H^2(\mathbb{A}_{1/\rho})}\Big]ds \\
&\qquad \leqslant Cr_0^2\|\mathbf{h}_0\|^2_{H^3(\mathbb{S}^1)} + C\|\!|\mathbf{h}|\!\|^3_T\,.
\end{aligned}$$

Estimate (106) then provides us with the inequality

$$\int_0^t \frac{1}{\log\rho}\|\mathbf{h}\|^2_{H^{3.5}(\mathbb{S}^1)}ds + \rho(t)^2\|\mathbf{h}(t)\|^2_{H^3(\mathbb{S}^1)} \leqslant Cr_0^2\|\mathbf{h}_0\|^2_{H^3(\mathbb{S}^1)} + C\|\!|\mathbf{h}|\!\|^3_T. \qquad (115)$$

REMARK 7.3 We note that the solution $\bar{p}$ satisfies the Taylor sign condition thanks to the inequality (6). Since we are considering only small perturbations of the linear solution, our nonlinear evolution is also satisfying the Taylor sign condition. In future work, we shall examine the large-perturbation regime wherein, the free-boundary can collide with itself as in [7].

### 7.5 *The stability of the Hele-Shaw flow with injection*

By the definition of the total norm and (115), we conclude that

$$|||\mathbf{h}|||_T^2 \leqslant C r_0^2 \|\mathbf{h}_0\|_{H^3(\mathbb{S}^1)}^2 + C |||\mathbf{h}|||_T^3$$

which, by Young's inequality, further suggests that

$$|||\mathbf{h}|||_T \leqslant C r_0 \|\mathbf{h}_0\|_{H^3(\mathbb{S}^1)} + C |||\mathbf{h}|||_T^2, \tag{116}$$

provided that condition (101) is valid. We then conclude Theorem 1.6 by the same argument used in Section 6.2.

*Acknowledgments.* AC was supported by the National Science Council (Taiwan) under grant 100-2115-M-008-009-MY3, and SS was supported by the National Science Foundation under grants DMS-1001850 and DMS-1301380, and by the Royal Society Wolfson Merit Award.

## REFERENCES

1. CHEN, X. The Hele-Shaw problem and area-preserving curve-shortening motions. *Arch. Rational Mech. Anal.* **123** (1993), 117–151. Zbl0780.35117 MR1219420
2. CHOI, S., JERISON, D., & KIM, I. Regularity for the one-phase Hele-Shaw problem from a Lipschitz initial surface. *Amer. J. Math.* **129** (2007), 527–582. Zbl1189.35384 MR2306045
3. CHOI, S., JERISON, D., & KIM, I. Local regularization of the one-phase Hele-Shaw flow. *Indiana Univ. Math. J.* **58** (2009), 2765–2804. Zbl1190.35229 MR2603767
4. CHOI, S., & KIM, I. C. Regularity of one-phase Stefan problem near Lipschitz initial data. *Amer. J. Math.* **132** (2010), 1693–1727. Zbl1231.35318 MR2766502
5. CONSTANTIN, P., & PUGH, M. Global solutions for small data to the Hele-Shaw problem. *Nonlinearity* **6** (1993), 393–415. Zbl0808.35104 MR1223740
6. COUTAND, D., & SHKOLLER, S. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.* **20** (2007), 829–930. Zbl1123.35038 MR2291920
7. COUTAND, D., & SHKOLLER, S. On the Finite-Time Splash and Splat Singularities for the 3-D Free-Surface Euler Equations. *Comm. Math. Phys.* **325** (2014), 143–183. Zbl1285.35071 MR3147437
8. DASKALOPOULOS, P., & LEE, K.-A. All time smooth solutions of the one-phase Stefan problem and the Hele-Shaw flow. *Comm. Partial Differential Equations* **29** (2004), 71–89. Zbl1099.35175 MR2038144
9. ELLIOTT, C. M., & JANOVSKÝ, V. A variational inequality approach to Hele-Shaw flow with a moving boundary. *Proc. Roy. Soc. Edinburgh Sect. A* **88** (1981), 93–107. Zbl0455.76043 MR0611303
10. ELLIOTT, C. M., & OCKENDON, J. R. *Weak and variational methods for moving boundary problems*, vol. 59 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass., 1982. Zbl0476.35080 MR0650455
11. ESCHER, J., & SIMONETT, G. Classical solutions for Hele-Shaw models with surface tension. *Adv. Differential Equations* **2** (1997), 619–642. Zbl1023.35527 MR1441859

12. ESCHER, J., & SIMONETT, G. Classical solutions of multidimensional Hele-Shaw models. *SIAM J. Math. Anal.* **28** (1997), 1028–1047. Zbl0888.35142 MR1466667

13. ESCHER, J., & SIMONETT, G. A center manifold analysis for the Mullins-Sekerka model. *J. Differential Equations* **143** (1998), 267–292. Zbl0896.35142 MR1607952

14. FRIEDMAN, A., & REITICH, F. Nonlinear stability of a quasi-static Stefan problem with surface tension: a continuation approach. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30** 2 (2001), 341–403. Zbl1072.35208 MR1895715

15. GUSTAFSSON, B. Applications of variational inequalities to a moving boundary problem for Hele-Shaw flows. *SIAM J. Math. Anal.* **16** (1985), 279–300. Zbl0605.76043 MR0777468

16. HOWISON, S. D. Cusp development in Hele-Shaw flow with a free surface. *SIAM J. Appl. Math.* **46** (1986), 20–26. Zbl0592.76042 MR0821438

17. PROKERT, G. Existence results for Hele-Shaw flow driven by surface tension. *European J. Appl. Math.* **9** (1998), 195–221. Zbl0919.35005 MR1630665

18. TAYLOR, M. E. *Partial Differential Equations I. Basic Theory*. Springer, 1997. Zbl0869.35002

19. TEMAM, R. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

20. VONDENHOFF, E. Large time behaviour of Hele-Shaw flow with injection or suction for perturbations of balls in $\mathbb{R}^N$. *IMA J. Appl. Math.* **76** (2011), 219–241. Zbl1219.35313 MR2781691