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On the Limit as the Density Ratio Tends to Zero for Two Perfect Incompressible Fluids Separated by a Surface of Discontinuity

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We study the asymptotic limit as the density ratio $\rho^-/\rho^+ \to 0$, where ρ^+ and ρ^- are the densities of two perfect incompressible 2-D/3-D fluids, separated by a surface of discontinuity along which the pressure jump is proportional to the mean curvature of the moving surface. Mathematically, the fluid motion is governed by the two-phase incompressible Euler equations with vortex sheet data. By rescaling, we assume the density ρ^+ of the inner *fluid is fixed, while the density ρ^- of the* outer *fluid is set to ϵ. We prove that solutions of the free-boundary Euler equations in vacuum are obtained in the limit as $\epsilon \to 0$.*

Keywords Euler equations; Surface tension; Two-phase flow; Vortex sheets; Water waves; Zero density limit.

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1. Introduction

1.1. *The Water Wave Problem*

A number of articles have recently appeared that focus on the analysis of the one-phase free-boundary incompressible Euler equations, in either irrotational form or with vorticity, in both 2-D and 3-D, and with or without surface tension effects on the free surface. See [6, 7, 10–14, 18, 19, 21] and the references therein. In irrotational form, the one-phase incompressible Euler equations with free-surface

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are often referred to as the *water wave equations* for the motion of the interface, since irrotationality decouples the motion of the liquid from that of the free-surface wave motion. The water wave equations typically model the motion of a liquid drop inside of air, or the waves on the surface of the ocean underneath the atmosphere (of course, air can be replaced with any other incompressible liquid with very small relative density). In particular, suppose that the density of the liquid is denoted by ρ^+ while the density of air (or the lighter liquid) by ρ^-. Even when $\rho^- \ll \rho^+$, the motion of the liquid-air system is more accurately modeled by the two-phase Euler equations, which in irrotational form lead to the equation of motion for *vortex sheets*.

The jump discontinuity in the tangential component of velocity across the material interface, which appears in the two-phase Euler model, is responsible for the ill-posedness of this system of PDE when surface tension effects are ignored (see [9, 20]). On the other hand, in the presence of surface tension, the two-phase system is well-posed. See [4] for existence and uniqueness of solutions to two-phase (rotational) Euler equations, and see [1–3] for the proof of well-posedness for the irrotational problem. Also, see [15] for an infinite-dimensional geometric approach to a priori estimates of the general problem. With surface tension included, the pressure experiences a jump discontinuity proportional to the mean curvature of the vortex sheet as we describe below. The two-phase system is a great deal more difficult to simulate computationally or study analytically, so it is of significant interest to rigorously establish the convergence of solutions to the two-phase equations (vortex sheets) to those of the one-phase model (water waves) in the limit as $\rho^-/\rho^+ \to 0$. The purpose of this paper is to prove this asymptotic result, without any irrotationality assumptions on the fluids. We state our results for the case that the space dimension is either 2 or 3, but we note that with additional regularity assumptions on the data, our results are valid for any space dimension great than 1.

1.2. *The Two-Phase Euler Equations in Eulerian Variables*

For $\mathfrak{n} = 2$ or 3, let $\mathcal{D} \subseteq \mathbb{R}^{\mathfrak{n}}$ denote an open, bounded set, which comprises the volume occupied by two incompressible and inviscid fluids with different densities. At the initial time $t = 0$, we let Ω^+ denote the volume occupied by the *inner* fluid with density ρ^+ and we let Ω^- denote the volume occupied by the *outer* fluid with density ρ^-. Mathematically, the sets Ω^+ and Ω^- denote two disjoint open bounded subsets of $\mathcal{D}$ such that $\overline{\mathcal{D}} = \overline{\Omega^+} \cup \overline{\Omega^-}$ and $\Omega^+ \cap \Omega^- = \emptyset$. The *material interface* at time $t = 0$ is given by $\Gamma := \overline{\Omega^+} \cap \overline{\Omega^-}$, and $\partial\mathcal{D} = \partial\Omega^- - \Gamma$.

Let $\Omega^+(t)$ and $\Omega^-(t)$ denote the time-dependent volumes of the inner and outer fluids, respectively, separated by the moving material interface $\Gamma(t)$. Let $u^\pm$ and $p^\pm$ denote the velocity field and pressure function, respectively, in $\Omega^\pm(t)$. Then the so-called *vortex sheet problem*, given by the solution of the incompressible Euler equations for the motion of two fluids separated by a *moving surface of discontinuity*, can be written as

$$\rho^\pm(u_t^\pm + u^\pm \cdot Du^\pm) + Dp^\pm = 0 \quad \text{in } \Omega^\pm(t), \tag{1.1a}$$

$$\operatorname{div} u^\pm = 0 \quad \text{in } \Omega^\pm(t), \tag{1.1b}$$

$$[p]_\pm = \sigma H \quad \text{on } \Gamma(t), \tag{1.1c}$$

$$[u \cdot n]_\pm = 0 \quad \text{on } \Gamma(t), \tag{1.1d}$$

$$u^- \cdot n = 0 \quad \text{on } \partial\mathcal{D}, \tag{1.1e}$$

$$u(0) = u_0 \quad \text{on } \{t = 0\} \times \mathcal{D}, \tag{1.1f}$$

$$\mathcal{V}(\Gamma(t)) = u^+(t) \cdot n(t), \tag{1.1g}$$

where $\mathcal{V}(\Gamma(t))$ denotes the speed of the moving interface $\Gamma(t)$ in the normal direction, and $n(t)$ denotes the outward-pointing unit normal to $\partial\Omega^+(t)$; thus, (1.1g) indicates that the vortex sheet $\Gamma(t)$ moves with the normal component of the fluid velocity. ρ^+ and ρ^- are the densities of the two fluids occupying $\Omega^+(t)$ and $\Omega^-(t)$, respectively, $H(t)$ is twice the mean curvature of $\Gamma(t)$, and $\sigma > 0$ is the surface tension parameter which we will henceforth set to one.

In [4], we proved the existence and uniqueness of the solutions to (1.1). See also [1–3] for the proof of well-posedness for the irrotational problem, and [15] for an infinite-dimensional geometric approach to a priori estimates of the general problem.

By rescaling, if necessary, we may assume that

$$\rho^+ = 1 \text{ and that } \rho^- = \epsilon \ll 1.$$

Letting $u_\epsilon^\pm$ denote the solutions of (1.1), the main objective of this article is to study the asymptotic behavior of the solutions $u_\epsilon^\pm$ as $\epsilon \to 0$.

1.3. *Notation*

1.3.1. Sobolev Norms on $\Omega^\pm$ and Γ. Let $H^s(\Omega^+)$ denote $H^s(\Omega^+; \mathbb{R})$ for scalar functions or $H^s(\Omega^+; \mathbb{R}^n)$ for vector fields, and let $H^s(\Omega^-)$ denote $H^s(\Omega^-; \mathbb{R})/\mathbb{R}$ for a scalar function with zero average or $H^s(\Omega^-; \mathbb{R}^n)$ for vector fields. We denote the $H^s(\Omega^\pm)$-norms by

$$\|w^+\|_{s,+} = \|w^+\|_{H^s(\Omega^+)} \quad \text{and} \quad \|w^-\|_{s,-} = \|w^-\|_{H^s(\Omega^-)}.$$

The $H^s(\Gamma)$- and $H^s(\partial\mathcal{D})$-norms are denoted by

$$|w^+|_s = \|w^+\|_{H^s(\Gamma)}, \quad |w^-|_s = \|w^-\|_{H^s(\Gamma)}, \quad \text{and} \quad |w|_{s,\partial\mathcal{D}} = \|w\|_{H^s(\partial\mathcal{D})}.$$

For simplicity, we also use $\|w\|_{s,\pm}^2$ and $|w|_{s,\pm}^2$ to denote $\|w^+\|_{s,+}^2 + \|w^-\|_{s,-}^2$ and $|w^+|_s^2 + |w^-|_s^2$, respectively, that is,

$$\|w\|_{s,\pm}^2 = \|w^+\|_{s,+}^2 + \|w^-\|_{s,-}^2,$$
$$|w|_{s,\pm}^2 = |w^+|_s^2 + |w^-|_s^2.$$

We also use $\langle \cdot, \cdot \rangle_{H^1(\Omega^+)}$, $\langle \cdot, \cdot \rangle_{H^1(\Omega^-)}$ and $\langle \cdot, \cdot \rangle_{H^{0.5}(\Gamma)}$ to denote the duality pairing between $H^1(\Omega^+)$ and $H^1(\Omega^+)'$, the duality pairing between $H^1(\Omega^-)$ and $H^1(\Omega^-)'$, and the duality paring between $H^{0.5}(\Gamma)$ and $H^{-0.5}(\Gamma)$, respectively.

1.3.2. Einstein Summation Convention. Repeated Latin indices are summed from 1 to n, while repeated Greek indices are summed from 1 to $n - 1$. For example,

$$f^\alpha g_\alpha := \sum_{\alpha=1}^{n-1} f^\alpha g_\alpha \quad \text{and} \quad f^i g_i := \sum_{i=1}^{n} f^i g_i.$$

1.3.3. The Tangential Derivative. Let $\{U_\ell\}_{\ell=1}^K$ denote an open covering of Γ, such that for each $\ell \in \{1, 2, \ldots, K\}$, with

$$\begin{aligned} V_\ell &= B(0, r_\ell), \text{ denoting the open ball of radius } r_\ell \text{ centered at the origin and,} \\ V_\ell^+ &= V_\ell \cap \{x_n > 0\}, \\ V_\ell^- &= V_\ell \cap \{x_n < 0\}, \end{aligned}$$

there exist for $s \geq 3$, H^s-class charts θ_ℓ which satisfy

$$\begin{aligned} \theta_\ell : V_\ell \to U_\ell \text{ is an } H^s \text{ diffeomorphism,} \\ \theta_\ell(V_\ell^+) &= U_\ell \cap \Omega^+, \\ \theta_\ell(V_\ell \cap \{x_n = 0\}) &= U_\ell \cap \Gamma. \end{aligned}$$

Next, for $L > K$, let $\{U_\ell\}_{\ell=K+1}^L$ denote a family of open balls of radius r_ℓ contained in Ω such that $\{U_\ell\}_{\ell=1}^L$ is an open cover of Ω, and let

$$\{\xi_\ell\}_{\ell=1}^L \text{ denote a } \mathscr{C}^\infty \text{ partition of unity subordinate to this covering of } \Omega.$$

We use $\bar{\partial}$ to denote the tangential derivative in $U_\ell \cap \Omega$. For a differentiable function f on Ω, the αth component of the tangential derivative of f is given by

$$f_{,\alpha} = \bar{\partial}_\alpha f = \frac{\partial}{\partial x_\alpha}[f \circ \theta_\ell] \circ \theta_\ell^{-1} = \left[(Df \circ \theta_\ell)\frac{\partial \theta_\ell}{\partial x_\alpha}\right] \circ \theta_\ell^{-1}.$$

We use $f_{,i}$ to denote the ith component of Df, where Df is the gradient of f, or

$$f_{,i} = \frac{\partial f}{\partial x_i}.$$

1.3.4. The Identity Map e. The identity map on $\mathbb{R}^n$ is denoted by e so that $e(x) = x$. For $\alpha = 1, 2$, we use the notation $e_{,\alpha}$ to denote the two tangent vectors to the reference material interface Γ; more specifically, in any local coordinate chart V_ℓ, $e,_\alpha$ denotes the tangent vectors $\frac{\partial \theta_\ell}{\partial x_\alpha}$. Note that

$$[(Df) \circ \theta_\ell] \cdot e_{,\alpha} = f_{,\alpha} \circ \theta_\ell \quad \text{or} \quad (f_{,j} \circ \theta_\ell) e_{,\alpha}^j = f_{,\alpha} \circ \theta_\ell.$$

1.3.5. H^s Norm of Γ. We defined the H^k-norm of Γ to be

$$|\Gamma|_k^2 := \sum_{\ell=1}^K \int_{\mathbb{R}}^{n-1} \xi_\ell |\partial_{\alpha_1 \cdots \alpha_k}^k \theta_\ell|^2 dx_1 \cdots dx_{n-1}.$$

The H^s-norm for any real $s \geq 0$ is defined by interpolation. We say that Γ is of class H^s (or $\Gamma \in H^s$) whenever $|\Gamma|_s < \infty$. The H^s-norm of $\partial\mathscr{D}$ is defined similarly.

1.3.6. Inner Products and Contractions. Given two vector v and w in $\mathbb{R}^n$, the inner product of v and w is denoted by $v \cdot w$, which in component is defined as

$$v \cdot w = v^i w_i = \sum_{i=1}^{n} v^i w_i.$$

For two matrices A and B, the contraction between A and B, denoted by $A : B$, is the trace of the product of A and B, which in component is defined as

$$A : B = \mathrm{Tr}(AB) = A_i^j B_j^i = \sum_{i,j=1}^{n} A_j^i B_i^j.$$

1.3.7. The Transpose of Matrices. Given any matrix $\mathscr{A}$, we use $\mathscr{A}^T$ to denote its transpose.

1.4. *The Arbitrary Lagrangian–Eulerian (ALE) Formulation*

Let η^+ denote the Lagrangian flow map of u^+ in Ω^+, that is,

$$\eta_t^+(x,t) = u^+(\eta^+(x,t),t) \quad \forall x \in \Omega^+, t > 0, \tag{1.2a}$$

$$\eta^+(x,0) = x \quad \forall x \in \Omega^+. \tag{1.2b}$$

By a theorem of Dacorogna and Moser [8], we can choose a volume preserving diffeomorphism $\overline{\psi}$ on Ω^- such that

$$\det(D\overline{\psi}) = 1 \quad \forall x \in \Omega^-,$$
$$\overline{\psi} = \eta^+ \quad \forall x \in \Gamma,$$
$$\overline{\psi} = e \quad \forall x \in \partial\mathscr{D}.$$

Furthermore, the following elliptic estimate holds:

$$\|\overline{\psi}\|_{4.5,-} \le C\big[|\eta^+|_4 + |\partial\mathscr{D}|_4\big].$$

We then define

$$\psi(x,t) = \begin{cases} \eta^+(x,t) & x \in \overline{\Omega^+}, \\ \overline{\psi}(x,t) & x \in \Omega^-, \end{cases}$$

Remark 1. We emphasize that ψ_t does not equal v^- in Ω^-; on the other hand,

$$\psi_t = v^+ \quad \text{on } \overline{\Omega^+}.$$

Set $v = u \circ \psi$, $q = p \circ \psi$, and let $A = (D\psi)^{-1}$. Using the ALE variables, equations (1.1) are written as

$$v_t^{+i} + A_i^k q_{,k}^+ = 0 \quad \text{in } (0,T) \times \Omega^+, \tag{1.3a}$$

$$\epsilon v_t^{-i} + \epsilon \mathrm{w}^j v_{,j}^{-i} + A_i^k q_{,k}^- = 0 \quad \text{in } (0,T) \times \Omega^-, \tag{1.3b}$$

$$A_i^j v_{,j}^{\pm i} = 0 \quad \text{in } (0,T)\times\Omega^\pm, \tag{1.3c}$$
$$q^+ - q^- = -\Delta_g \eta \cdot n \quad \text{on } (0,T)\times\Gamma, \tag{1.3d}$$
$$v^+\cdot n = v^-\cdot n \quad \text{on } (0,T)\times\Gamma, \tag{1.3e}$$
$$v^-\cdot N = 0 \quad \text{on } \partial\mathscr{D}, \tag{1.3f}$$
$$(\psi(t), v(t), \Omega^\pm(t))_{t=0} = (e, u_0, \Omega^\pm), \tag{1.3g}$$

where $e(x) = x$ denotes the identity map on $\mathscr{D}$, $\mathrm{w} = A(v^- - \psi_t)$, and $n(t) := n(\psi(t))$ denotes the outward-point unit normal to $\partial\Omega^+(t)$ and evaluated at the point $\psi(t)$. With N denoting the outward-point unit normal to $\partial\Omega^+$ at $t=0$, we have the identity

$$n(\psi(t)) = \frac{A^T N}{|A^T N|}.$$

1.5. *The Higher-Order Energy Function*

With $\mathscr{S}(t)$ denoting the surface area of the vortex sheet $\Gamma(t)$, the physical energy function is given by $\|u^+\|^2_{L^2(\Omega^+(t))} + \epsilon\|u^-\|^2_{L^2(\Omega^-(t))} + 2\sigma\mathscr{S}(t)$. While the physical energy is exactly conserved, it is much too weak to provide the necessary *a priori* control to pass to the limit as $\epsilon \to 0$. As such, we define the higher-order energy function $\mathscr{E}$

$$\mathscr{E}(t) = |\bar\partial\eta^+\cdot N|^2_{3,+} + \|v\|^2_{3,\pm} + \|v_t\|^2_{1.5,\pm} + \|v_{tt}^+\|^2_{0,+} + \epsilon\|v_{tt}^-\|^2_{0,-}.$$

Note that only $\epsilon\|v_{tt}^-\|^2_{0,-}$ has the asymptotic scaling parameter ϵ.

1.6. *The Regularity of the Solution to* (1.1)

With $\epsilon = \rho^-/\rho^+$, the following theorem is the main result in [4].

Theorem (Well-posedness of (1.1))**.** *Suppose that $\sigma > 0$, and that $\Gamma := \Gamma(0)$ is of class H^4, $\partial\mathscr{D}$ is of class H^3, and $u_0^\pm \in H^3(\Omega^\pm)$. Then, for all $\epsilon > 0$, there exists $T_\epsilon > 0$, and a solution $(u^\pm(t), p^\pm(t), \Omega^\pm(t))$ of* (1.1) *with $u^\pm \in L^\infty(0,T; H^3(\Omega^\pm(t)))$, $p^\pm \in L^\infty(0,T; H^{2.5}(\Omega^\pm(t)))$, and $\Gamma(t) \in H^4$. The solution is unique if $u_0^\pm \in H^{4.5}(\Omega^\pm)$ and $\Gamma \in H^{5.5}$.*

Note that the time of existence T_ϵ depends crucially upon ϵ, and that *a priori* T_ϵ may approach zero as $\epsilon \to 0$.

1.7. *Main Result*

Let $\Omega(t) = \Omega^+(t)$, and let U denote the solution of the one-phase free-surface incompressible Euler equations in vacuum, satisfying

$$U_t + U\cdot DU + DP = 0 \quad \text{in } \Omega(t), \tag{1.4a}$$
$$\operatorname{div} U = 0 \quad \text{in } \Omega(t), \tag{1.4b}$$
$$P = H \quad \text{on } \Gamma(t), \tag{1.4c}$$
$$U\cdot n = 0 \quad \text{on } \Gamma(t), \tag{1.4d}$$

$$u(0) = u_0 \quad \text{on } \{t = 0\} \times \Omega, \tag{1.4e}$$

$$\mathcal{V}(\Gamma(t)) = U(t) \cdot n(t). \tag{1.4f}$$

Theorem 1.1 (Main Theorem). *Let $\epsilon = \rho^-/\rho^+$, and for $\epsilon > 0$, let u_ϵ^+ denote the sequence of the solution to (1.1) in the inner phase $\Omega^+(t)$. Suppose that $u_0^+ = u_0|_{\Omega^+} \in H^3(\Omega^+)$ and $u_0^- = u_0|_{\Omega^-} \in H^3(\Omega^-)$ satisfying $u_0^+ \cdot N = u_0^- \cdot N$ on Γ, where Γ is of class $H^{4.5}$ and $\partial\mathcal{D}$ is smooth. Then there exists $T > 0$, independent of ϵ, such that the solution $u_\epsilon^+ \circ \eta_\epsilon$ to (1.1) converges weakly to $U \circ \eta$ in $L^2(0, T; H^3(\Omega^+))$ as $\epsilon \to 0$, where η_ϵ and η are flows of u_ϵ^+ and U, respectively.*

Remark 2. Note that u_ϵ *a priori* only exists on the ϵ-dependent time interval $(0, T_\epsilon)$; however, the main theorem shows that T_ϵ is in fact independent of ϵ, and that u_ϵ exists on an ϵ-independent interval $(0, T)$ for all $\epsilon > 0$.

1.8. *The Structure of the Proof and Outline of the Paper*

The proof of Theorem 1.1 consists of several steps that we describe as follows. In Section 2, we review some well-known inequalities that we use throughout our analysis. In Section 3, we establish estimates for the time derivatives of velocity and pressure, evaluated at time $t = 0$. Section 4 is devoted to the derivation of ϵ-independent estimates for our two-phase system. The fundamental difficulty resides in the estimates for the normal and tangential components of v_{tt}^-, which are founded on improved elliptic estimates (with respect to our estimates in [4]) for the pressure functions. Finally, in Section 5, we pass to the limit as $\epsilon \to 0$ and establish our main result.

2. Preliminary Results

2.1. *The Trace of the Normal Component of a Vector Field*

A vector $u \in L^2(\mathcal{O})$ with $\operatorname{div} u \in H^1(\mathcal{O})'$ has a *normal trace* $u \cdot N \in H^{-0.5}(\partial\mathcal{O})$, where N is the unit normal to the surface $\partial\mathcal{O}$, with the estimate

$$\|u \cdot N\|^2_{H^{-0.5}(\partial\mathcal{O})} \le C\big[\|u\|^2_{L^2(\mathcal{O})} + \|\operatorname{div} u\|^2_{H^1(\mathcal{O})'}\big], \tag{2.1}$$

where C depends on $|\Gamma|_s$ for all $s > \frac{n+2}{2}$ (see, for example, [17]).

By the Piola identity, $A_i^k{}_{,k} = 0$ (since $\det D\psi = 1$), and the identity $A^T N = \sqrt{g} n$, letting $u^j = A_i^j w^i$ in (2.1) yields the Lagrangian normal trace estimate

$$\|w \cdot n\|^2_{H^{-0.5}(\partial\mathcal{O})} \le C\big[\|w\|^2_{L^2(\mathcal{O})} + \|A_i^j w^i_{,j}\|^2_{H^1(\mathcal{O})'}\big], \tag{2.2}$$

where $\mathcal{O}$ is either Ω^+ or Ω^-.

2.2. *The Hodge Decomposition Elliptic Estimate*

Proposition 2.1. *For $r \ge 2.5$, let $\mathcal{O}$ be H^r domain, that is, $\partial\mathcal{O}$ is of class $H^{r-0.5}$. If $w \in L^2(\mathcal{O}; \mathbb{R}^3)$ with $\operatorname{curl} w \in H^{r-1}(\mathcal{O})$, $\operatorname{div} w \in H^{r-1}(\mathcal{O})$, and $\bar\partial w \cdot N \in H^{r-1.5}(\partial\mathcal{O})$, then there exists a constant C depending on $|\partial\mathcal{O}|_{r-0.5}$ such that*

$$\|w\|_{H^r(\mathcal{O})} \le C(|\partial\mathcal{O}|_{r-0.5})\big[\|w\|_{L^2(\mathcal{O})} + \|\operatorname{curl} w\|_{H^{r-1}(\mathcal{O})} + \|\operatorname{div} w\|_{H^{r-1}(\mathcal{O})} + \|\bar\partial w \cdot N\|_{H^{r-1.5}(\partial\mathcal{O})}\big]. \tag{2.3}$$

This estimate is well-known and follows from the identity $-\Delta F = \operatorname{curl}\operatorname{curl} F - D\operatorname{div} F$; a convenient reference is Taylor [16].

2.3. *The Curl and Divergence Estimates of η, v and v_t*

Exactly following Section 10 in [6], we have the following

Lemma 2.1. *The quantities $D\operatorname{div}\eta^+$, $\operatorname{div} v^\pm$, $\operatorname{div} v_t^\pm$ and $D\operatorname{curl}\eta^+$, $\operatorname{curl} v^\pm$, $\operatorname{curl} v_t^\pm$ satisfy the following estimates*:

$$\begin{aligned}
&\|\operatorname{div} v_t^\pm\|_{0.5,\pm}^2 + \|\operatorname{curl} v_t^\pm\|_{0.5,\pm}^2 + \|\operatorname{div} v^\pm\|_{2,\pm}^2 \\
&\quad + \|\operatorname{curl} v^\pm\|_{2,\pm}^2 + \|D\operatorname{div}\eta^+\|_{2.5,+}^2 + \|D\operatorname{curl}\eta^+\|_{2.5,+}^2 \\
&\quad \leq C_\delta \mathscr{M}_0 + C_\delta T \mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right) + \delta \sup_{t\in[0,T]} \mathscr{E}(t),
\end{aligned} \tag{2.4}$$

where $\delta > 0$ is taken sufficiently small, and P denotes a polynomial function of its argument.

2.4. *A Polynomial-Type Inequality*

For a constant $M \geq 0$, suppose that $f(t) \geq 0$, $t \mapsto f(t)$ is continuous, and

$$f(t) \leq M + CtP(f(t)), \tag{2.5}$$

where P denotes a polynomial function, and C is a generic constant. Then for t taken sufficiently small, we have the bound

$$f(t) \leq 2M.$$

This type of inequality, which we introduced in [5], can be viewed as a generalization of standard nonlinear Gronwall inequalities.

2.5. *Differentiating the Matrix A*

In this subsection we list a very useful identity here concerning the differentiation of the cofactor matrix A for reference. Let δ be a differential operator such as ∂_t, $\bar\partial$ or D, then

$$\delta A_i^j = -A_r^j \delta\psi^r_{,s} A_i^s. \tag{2.6}$$

For example, when $\delta = \partial_t$,

$$(A_i^j)_t = -A_r^j \psi^r_{t,s} A_i^s.$$

3. Estimates for Velocity, Pressure, and their Time Derivatives at Time $t = 0$

3.1. *Estimates for the Initial Data*

We require estimates for the time derivatives of the velocity and pressure at $t = 0$. As in [4], we use w_1, w_2, q_0 and q_1 to denote $v_t(0)$, $v_{tt}(0)$, $q(0)$ and $q_t(0)$, respectively. Following [4], estimates for q_i can be obtained by analyzing certain elliptic equations, and estimates for w_i are obtained by letting $t = 0$ in (1.3a) and (1.3b). The estimates obtained in [4] are density dependent. In particular, w_i and q_i satisfy

$$\|w_1^+\|_{2,+}^2 + \|q_0^+\|_{3,+}^2 + \epsilon\|w_1^-\|_{2,-}^2 + \frac{1}{\epsilon}\|q_0^-\|_{3,-}^2 \leq C\mathscr{P}(\|u_0\|_{3,\pm}^2, |\Gamma|_{4.5}^2) \tag{3.1}$$

and

$$\|w_2^+\|_{0,+}^2 + \epsilon\|w_2^-\|_{0,-}^2 + \|q_1^+\|_{1,+}^2 + \frac{1}{\epsilon}\|q_1^-\|_{1,-}^2 \leq C\mathscr{P}(\|u_0\|_{3,\pm}^2, |\Gamma|_{4.5}^2), \tag{3.2}$$

where $\mathscr{P}$ is some polynomial of its variables.

However, in order to obtain ϵ-independent estimates, we require ϵ-independent bounds for q_i^- and w_i^-. Indeed, we have the following

Proposition 3.1. *Given $u_0^+ \in H^3(\Omega^+)$, $u_0^- \in H^3(\Omega^-)$, and $\Gamma \in H^{4.5}$, then*

$$\begin{aligned}\|w_1\|_{2,\pm}^2 + \|w_2\|_{0,\pm}^2 + \|q_0^+\|_{3,+}^2 + \|q_1^+\|_{1,+}^2 \\ + \frac{1}{\epsilon^2}\big[\|q_0^-\|_{2,-}^2 + \|q_1^-\|_{0,-}^2\big] \leq C\mathscr{P}(\|u_0\|_{3,\pm}^2, |\Gamma|_{4.5}^2).\end{aligned} \tag{3.3}$$

Proof. We note that the estimates for w_i^+ and q_i^+ follow from (3.1) and (3.2), so it suffices to obtain estimates for w_i^- and q_i^-. We estimate q_0^- first.

Taking the Lagrangian divergence of (1.3b), by the Lagrangian divergence-free condition $A_i^j v_{,j}^{-i} = 0$, we obtain

$$\epsilon\big[-A_{it}^j v_{,j}^{-i} - A_{i,\ell}^j v_{,j}^{-i}\mathrm{w}^\ell + A_i^j \mathrm{w}_{,j}^\ell v_{,\ell}^{-i}\big] + A_i^j(A_i^k q_{,k}^-)_{,j} = 0.$$

Using (2.6) and $\mathrm{w} = A(v^- - \psi_t)$ in the equality above at $t = 0$, as well as restricting (1.3b) on $\partial\Omega^-$ in the normal direction at $t = 0$, we find that q_0^- satisfies

$$\Delta q_0^- = -\epsilon(Du_0^-)^T : (Du_0^-) \quad \text{in } \Omega^-, \tag{3.4a}$$

$$\frac{\partial q_0^-}{\partial N} = -\epsilon w_1^- \cdot N - \epsilon(\mathrm{w}_0 \cdot Du_0^-) \cdot N \quad \text{on } \Gamma, \tag{3.4b}$$

$$\frac{\partial q_0^-}{\partial N} = -\epsilon(\mathrm{w}_0 \cdot Du_0^-) \cdot N \quad \text{on } \partial\mathscr{D}. \tag{3.4c}$$

By (1.3e),

$$w_1^- \cdot N = w_1^+ \cdot N + g_0^{\gamma\delta}(u_{0,\gamma}^+ \cdot N)[(u_0^+ - u_0^-) \cdot e_{,\delta}]. \tag{3.5}$$

Therefore, by elliptic regularity,

$$\begin{aligned}\|q_0^-\|_{3,-}^2 &\le C\epsilon^2\big[\|u_{0,j}^{-i}u_{0,i}^{-j}\|_{1,-}^2 + |w_1^+\cdot N|_{1.5}^2 + |g_0^{\gamma\delta}(u_{0,\gamma}^+\cdot N)(u_0^+ - u_0^-)\cdot e_{,\delta}|_{1.5}^2 \\ &\quad + |(\mathrm{w}_0\cdot Du_0^-)\cdot N|_{1.5,-}^2 + |(\mathrm{w}_0\cdot Du_0^-)\cdot N|_{1.5,\partial\mathscr{D}}^2\big] \\ &\le C\epsilon^2\mathscr{P}(\|u_0\|_{3,\pm}^2, |\Gamma|_{4.5}^2). \end{aligned} \tag{3.6}$$

By (3.6) and (1.3b), we also obtain an ϵ-independent estimate for w_1^-:

$$\|w_1^-\|_{2,-}^2 \le C\Big[\|\mathrm{w}_0\cdot Du_0^-\|_{2,-}^2 + \frac{1}{\epsilon^2}\|q_0^-\|_{3,-}^2\Big] \le C\mathscr{P}(\|u_0\|_{3,\pm}^2, |\Gamma|_{4.5}^2). \tag{3.7}$$

Similarly, since

$$w_2^-\cdot N = w_2^+\cdot N + 2(w_1^+ - w_1^-)\cdot n_t(0) + (u_0^+ - u_0^-)\cdot n_{tt}(0),$$

by (3.7) we find that

$$\begin{aligned}|w_2^-\cdot N|_{-0.5}^2 &\le C\big[|w_2^+\cdot N|_{-0.5}^2 + |w_1^+ - w_1^-|_{1.5}^2|n_t(0)|_0^2 + |u_0^+ - u_0^-|_{1.5}^2|n_{tt}(0)|_{-0.5}^2\big] \\ &\le C\mathscr{P}(\|u_0\|_{3,\pm}^2, |\Gamma|_{4.5}^2).\end{aligned}$$

Hence by considering the elliptic problem for q_1^- (see [4, p. 14] or letting $t=0$ in (4.11a) for the precise equations), the elliptic regularity implies

$$\|q_1^-\|_{1,-}^2 \le C\epsilon^2\mathscr{P}(\|u_0\|_{3,\pm}^2, |\Gamma|_{4.5}^2).$$

Time-differentiating (1.3b) and setting $t=0$ then yields the estimate

$$\|w_2^-\|_{0,-}^2 \le C\mathscr{P}(\|u_0\|_{3,\pm}^2, |\Gamma|_{4.5}^2).$$

□

Henceforth, we let $\mathscr{M}_0$ denote a constant depending on $\|u_0\|_{3,\pm}$ and $|\Gamma|_{4.5}$. Therefore, (3.3) implies

$$\|w_1\|_{2,\pm}^2 + \|w_2\|_{0,\pm}^2 + \|q_0^+\|_{3,+}^2 + \|q_1^+\|_{1,+}^2 + \frac{1}{\epsilon^2}\big[\|q_0^-\|_{3,-}^2 + \|q_1^-\|_{1,-}^2\big] \le \mathscr{M}_0. \tag{3.8}$$

In the later discussion, we also need the lower order estimates for $q_0^\pm$ and $w_1^\pm$. Instead of Proposition 3.1, we have the following

Proposition 3.2. *Given $u_0^+\in H^{1.5}(\Omega^+)$, $u_0^-\in H^{1.5}(\Omega^-)$, and $\Gamma\in H^3$, then*

$$\|w_1\|_{0,\pm}^2 + \|q_0^+\|_{1,+}^2 + \frac{1}{\epsilon^2}\|q_0^-\|_{1,-}^2 \le C\mathscr{P}(\|u_0\|_{1.5,\pm}^2, |\Gamma|_3^2). \tag{3.9}$$

Proof. The parts $\|w_1^+\|_{0,+}^2$ and $\|q_0^+\|_{1,+}^2$ in (3.9) follows from [4] by first solving

$$\begin{aligned}\Delta q_0^+ &= -u_{0,j}^{+i}u_{0,i}^{+j} \quad \text{in } \Omega^+,\\ \frac{1}{\epsilon}\Delta q_0^- &= -u_{0,j}^{-i}u_{0,i}^{-j} \quad \text{in } \Omega^-,\end{aligned}$$

$$\begin{aligned} q_0^+ - q_0^- &= H_0 \quad \text{on } \Gamma, \\ \frac{\partial q_0^+}{\partial N} - \frac{1}{\epsilon}\frac{\partial q_0^-}{\partial N} &= \left[(u_0^- - u_0^+)\cdot e_{,\beta} g_0^{\alpha\beta} u_{0,\alpha}^+ + (\mathrm{w}_0 \cdot Du_0^-)\right]\cdot N \quad \text{on } \Gamma, \\ \frac{\partial q_0^-}{\partial N} &= -\epsilon(\mathrm{w}_0 \cdot Du_0^-)\cdot N \quad \text{on } \partial\mathcal{D}, \end{aligned}$$

to obtain that

$$\begin{aligned} \|q_0^+\|_{1,+}^2 + \frac{1}{\epsilon}\|q_0^-\|_{1,-}^2 &\le C\big[\|u_{0,j}^{+i}u_{0,i}^{+j}\|_{L^{6/5}(\Omega^+)}^4 + \|u_{0,j}^{-i}u_{0,i}^{-j}\|_{L^{6/5}(\Omega^-)}^4 + |H_0|_{0.5}^2 \\ &\quad + |[(u_0^- - u_0^+)\cdot e_{,\beta} g_0^{\alpha\beta} u_{0,\alpha}^+ + (\mathrm{w}_0\cdot Du_0^-)]\cdot N|_{-0.5}^2 \\ &\quad + |(\mathrm{w}_0\cdot Du_0^-)\cdot N|_{-0.5,\partial\mathcal{D}}^2\big] \\ &\le C\mathcal{P}(\|u_0\|_{1.5,\pm}^2, |\Gamma|_3^2), \end{aligned}$$

and the estimate for w_1^+ follows from the Euler equations. Then we test (3.4a) against q_0^- to find that

$$\begin{aligned} \|Dq_0^-\|_{0,-}^2 &= -\int_{\Omega^-} \epsilon u_{0,j}^{-i}u_{0,i}^{-j} q_0^- \, dx + \int_{\Gamma\cup\partial\mathcal{D}} q_0^- \frac{\partial q_0^-}{\partial N} dS \\ &\le \epsilon\|u_{0,j}^{-i}u_{0,i}^{-j}\|_{L^{6/5}(\Omega^-)}\|q_0^-\|_{L^6(\Omega^-)} - \epsilon\int_\Gamma q_0^-(w_1^- + \mathrm{w}_0\cdot Du_0^-)\cdot N dS \\ &\quad - \epsilon\int_{\partial\mathcal{D}} q_0^-(\mathrm{w}_0\cdot Du_0^-)\cdot N dS \\ &\le C_\delta \epsilon^2 \mathcal{P}(\|u_0\|_{1.5,\pm}^2, |\Gamma|_3^2) + \delta\|q_0^-\|_{1,-}^2, \end{aligned}$$

where we use (3.5) to estimate $w_1^- \cdot N$ in terms of $u_0^\pm$ and w_1^+. By Poincare's inequality, we find that (3.9) holds for $\|q_0^-\|_{1,-}^2$ and therefore, by the Euler equations, for $\|w_1^-\|_{0,-}$ as well. □

3.2. *Basic Assumptions on Bounds*

We assume that we have a sufficiently smooth solution v^+, such that on the time interval $[0, T]$,

$$\begin{cases} \|\eta^+ - e\|_{3,+}^2 \le \dfrac{1}{2}, & \|D\eta^+\|_{2,+}^2 \le \mathrm{n}|\Omega^+|^2 + 1, \\ \|v^+\|_{1.5,+}^2 \le \|u_0^+\|_{1.5,+}^2 + 1, & \|v^-\|_{1.5,-}^2 \le \|u_0^-\|_{1.5,-}^2 + 1, \\ \|v_t^+\|_{0,+}^2 \le \|w_1^+\|_{0,+}^2 + 1, & \|\mathrm{w}\|_{1.5,-}^2 \le 2\|\mathrm{w}_0\|_{1.5,-}^2 + 1, \\ \|v_t^-\|_{0,-}^2 \le \mathcal{P}(\|u_0\|_{1.5,\pm}^2, |\Gamma|_3^2), & \end{cases}$$

where $\mathcal{P}$ is a polynomial of its variable. Verification of these assumptions, except for $\|v_t^-\|_{0,-}$, will follow from the fundamental theorem of calculus, once our energy estimates are completed.

In the following, we allow our generic constant C to depend on the right-hand sides of these inequalities. Given the estimate (3.9), the constant C depends only on the measure of Ω^+, $\|u_0\|_{1.5,\pm}^2$ and $|\Gamma|_3^2$.

3.3. *The Estimates for $w = A(v^- - \psi_t)$ and w_t*

By the fundamental theorem of Calculus,

$$\|\mathrm{w}(t)\|_{1.5,-}^2 \le 2\big[\|\mathrm{w}(0)\|_{1.5,-}^2 + t\int_0^t \|\mathrm{w}_t(s)\|_{1.5,-}^2 ds\big]. \tag{3.10}$$

Since $\mathrm{w} = A(v^- - \psi_t)$,

$$\mathrm{w}_t^j = -A_r^j \psi_{t,s}^r A_\ell^s (v^{-\ell} - \psi_t^\ell) + A_\ell^j (v_t^{-\ell} - \psi_{tt}^\ell) = -A_r^j \psi_{t,s}^r \mathrm{w}^s + A_\ell^j (v_t^{-\ell} - \psi_{tt}^\ell);$$

hence

$$\|\mathrm{w}_t\|_{0,-}^2 \le \|A\|_{L^\infty(\Omega^-)}^2 \big[\|D\psi_t\|_{L^3(\Omega^-)}^2 \|\mathrm{w}\|_{L^6(\Omega^-)}^2 + 2(\|v_t^-\|_{0,-}^2 + \|\psi_{tt}\|_{0,-}^2)\big] \le C$$

and

$$\|\mathrm{w}_t\|_{1.5,-}^2 \le C\big[\|\psi_t\|_{2.75,-}^2 \|\mathrm{w}\|_{1.5,-}^2 + \|v_t^-\|_{1.5,-}^2 + \|\psi_{tt}\|_{1.5,-}^2\big] \le C \sup_{t\in[0,T]} \mathscr{E}(t).$$

4. The ϵ-Independent Estimates

4.1. *Estimates for the Pressure and $v_{tt}^\pm \cdot n$*

Proposition 4.1. *Given* $\mathscr{E}(t)$ *defined in Section* 1.5, *the solution* $v^\pm$ *of* (1.3) *satisfies the following estimate*:

$$|v_{tt}\cdot n|_{-0.5,\pm}^2 + \|q^+\|_{2.5,+}^2 + \|q_t^+\|_{1,+}^2 + \frac{1}{\epsilon^2}\big[\|q^-\|_{2.5,-}^2 + \|q_t^-\|_{1,-}^2\big] \le C \sup_{t\in[0,T]} \mathscr{E}(t). \tag{3.10}$$

Proof. The proof consists of four steps.

Step 1 (Estimates for q^-). Since q^- satisfies

$$\begin{aligned} A_i^j (A_i^k q_{,k}^-)_{,j} &= -\epsilon\big[(A_i^j)_t v_{,j}^{-i} + (A_i^j)_{,\ell} \mathrm{w}^\ell v_{,j}^{-i} - A_i^j \mathrm{w}_{,j}^\ell v_{,\ell}^{-i}\big] \quad \text{in } \Omega^-, \\ A_i^j A_i^k q_{,k}^- N_j &= \epsilon\big[-v_t^{-i} - \mathrm{w}^\ell v_{,\ell}^{-i}\big] A_i^j N_j \quad \text{on } \Gamma, \\ q_{,k}^- N_k &= -\epsilon \mathrm{w}^\ell v_{,\ell}^{-i} N_i \quad \text{on } \partial\mathscr{D}, \end{aligned}$$

elliptic regularity shows that

$$\begin{aligned} \frac{1}{\epsilon^2}\|q^-\|_{2.5,-}^2 \le C\big[&\|AD\mathrm{w}Dv^-\|_{0.5,-}^2 + \|\mathrm{w}DADv^-\|_{0.5,-}^2 + \|A_t Dv^-\|_{0.5,-}^2 \\ &+ |v_t^{-i} n^i|_1 + |(\mathrm{w}\cdot Dv^-)\cdot n|_1 + |(\mathrm{w}\cdot Dv^-)\cdot N|_{1,\partial\mathscr{D}}\big]. \end{aligned}$$

We first estimate $\|AD\mathrm{w}Dv^-\|_{0.5,-}^2$. This requires interpolation, so we first estimate $\|AD\mathrm{w}Dv^-\|_{0,-}^2$ and $\|AD\mathrm{w}Dv^-\|_{1,-}^2$. It is easy to see that

$$\|AD\mathrm{w}Dv^-\|_{0,-}^2 \le \|A\|_{L^\infty(\Omega^-)}^2 \|D\mathrm{w}\|_{L^6(\Omega^-)}^2 \|Dv^-\|_{L^3(\Omega^-)}^2 \le C\big[\|v^+\|_{2,+}^2 + \|v^-\|_{2,-}^2\big]$$

and that

$$\begin{aligned}\|D(AD\mathrm{w}Dv^-)\|_{0,-}^2 &\le C\big[\|DAD\mathrm{w}Dv^-\|_{0,-}^2 + \|AD^2\mathrm{w}Dv^-\|_{0,-}^2 + \|AD\mathrm{w}D^2v^-\|_{0,-}^2\big]\\ &\le C\big[\|v^+\|_{3,+}^2 + \|v^-\|_{3,-}^2\big];\end{aligned}$$

hence, by interpolation,

$$\|\mathrm{w}DADv^-\|_{0.5,-}^2 \le C\big[\|v^+\|_{2.5,+}^2 + \|v^-\|_{2.5,-}^2\big].$$

Similarly, by interpolation we find that

$$\|\mathrm{w}DADv^-\|_{0.5,-}^2 + \|A_tDv^-\|_{0.5,-}^2 \le C\big[\|v^+\|_{2.5,+}^2 + \|v^-\|_{2.5,-}^2\big].$$

It is also easy to see that $|v_t^- \cdot n|_1 \le C\|v_t^-\|_{1.5}$ and that

$$|(\mathrm{w}\cdot Dv^-)\cdot n|_1^2 + |(\mathrm{w}\cdot Dv^-)\cdot N|_{1,\partial\mathscr{D}}^2 \le C\|v^-\|_3^2.$$

It follows that

$$\frac{1}{\epsilon^2}\|q^-\|_{2.5,-}^2 \le C\sup_{t\in[0,T]}\mathscr{E}(t). \tag{4.2}$$

Step 2 (Uniform Bounds for $v_{tt}^\pm \cdot n$). Letting $w = v_{tt}^+$ in (2.2),

$$|v_{tt}^+\cdot n|_{-0.5} \le C\big[\|v_{tt}^+\|_{0,+} + \|A_i^j v_{tt,j}^{+i}\|_{H^1(\Omega^+)'}\big].$$

By the incompressibility condition (1.3c),

$$A_i^j v_{tt,j}^{+i} = -(A_i^j)_{tt}v_{,j}^{+i} - 2(A_i^j)_t v_{t,j}^{+i}. \tag{4.3}$$

Let $f\in H^1(\Omega^+)$; then

$$\langle (A_i^j)_{tt}v_{,j}^{+i}, f\rangle_{H^1(\Omega^+)} = -\int_{\Omega^+}(A_i^j)_{tt}v^{+i}f_{,j}dx + \int_\Gamma \big(\sqrt{g}n^i\big)_{tt}v^{+i}fdS. \tag{4.4}$$

Taking the supremum over all $f\in H^1(\Omega^+)$ with $\|f\|_{1,+}=1$, we find that

$$\begin{aligned}\|(A_i^j)_{tt}v_{,j}^{+i}\|_{H^1(\Omega^+)'}^2 &\le C\big[\|(A_i^j)_{tt}v^{+i}\|_{0,+}^2 + |(\sqrt{g}n)_{tt}\cdot v^+|_{-0.5}^2\big]\\ &\le C\big[(\|Dv_t^+\|_{L^3(\Omega^+)}^2 + \|Dv^+\|_{L^6(\Omega^+)}^4)\|v^+\|_{L^6(\Omega^+)}^2 + |v_t^+|_{0.5}^2 + |v^+|_{0.5}^2|v^+|_{2.25}^2\big]\\ &\le C\big[\|v_t^+\|_{1.5,+}^2 + \|v^+\|_{3,+}^2\big] \le C\sup_{t\in[0,T]}\mathscr{E}(t).\end{aligned}$$

Similarly,

$$\langle (A_i^j)_t v_{t,j}^{+i}, f\rangle_{H^1(\Omega^+)} = -\int_{\Omega^+}(A_i^j)_t v_t^{+i}f_{,j}dx + \int_\Gamma(\sqrt{g}n^i)_t v_t^{+i}fdS$$

which implies

$$\begin{aligned}\|(A_i^j)_t v_{t,j}^+\|_{H^1(\Omega^+)'}^2 &\le C\big[\|(A_i^j)_t v_t^{+i}\|_{0,+}^2 + |(\sqrt{g}n)_t\cdot v_t^+|_{-0.5}^2\big]\\ &\le C\big[\|Dv^+\|_{L^3(\Omega^+)}^2\|v_t^+\|_{L^6(\Omega^+)}^2 + \|v^+\|_{W^{1,4}(\Gamma)}^2\|v_t^+\|_{L^4(\Gamma)}^2\big]\end{aligned}$$

$$\leq C\big[\|v_t^+\|_{1,+}^2 + \|v^+\|_{1.5,+}^{4/3}\|v^+\|_{3,+}^{2/3}\|v_t^+\|_{0,+}^{2/3}\|v_t^+\|_{1.5,+}^{4/3}\big]$$
$$\leq C\big[\|v_t^+\|_{1,+}^2 + \|v^+\|_{1.5,+}^2\|v_t^+\|_{1.5,+}^2 + \|v_t^+\|_{0,+}^2\|v^+\|_{3,+}^2\big]$$
$$\leq C \sup_{t\in[0,T]} \mathscr{E}(t). \tag{4.5}$$

Therefore,

$$\|A_i^j v_{tt,j}^+\|_{H^1(\Omega^+)'}^2 \leq C \sup_{t\in[0,T]} \mathscr{E}(t) \tag{4.6}$$

and hence

$$|v_{tt}^+ \cdot n|_{-0.5}^2 \leq C \sup_{t\in[0,T]} \mathscr{E}(t).$$

The ϵ-independent estimate for $v_{tt}^- \cdot n$ is obtained using a different argument. By (1.3e),

$$v_{tt}^- \cdot n = v_{tt}^+ \cdot n + (v^+ - v^-) \cdot n_{tt} + 2(v_t^+ - v_t^-) \cdot n_t;$$

hence

$$|v_{tt}^- \cdot n|_{-0.5}^2 \leq C\big[|v_{tt}^+ \cdot n|_{-0.5}^2 + |(v^+ - v^-) \cdot n_{tt}|_{-0.5}^2 + |(v_t^+ - v_t^-) \cdot n_t|_{-0.5}^2\big].$$

We claim that for any $f \in H^{1.25}(\Gamma)$ and $g \in H^{-0.5}(\Gamma)$,

$$|fg|_{-0.5} \leq C|f|_{1.25}|g|_{-0.5}. \tag{4.7}$$

To see this, note that

$$|fg|_{-0.5} = \sup_{|\phi|_{0.5}=1} |\langle fg, \phi\rangle_{H^{0.5}(\Gamma)}| = \sup_{|\phi|_{0.5}=1} |\langle g, f\phi\rangle_{H^{0.5}(\Gamma)}| \leq |g|_{-0.5} \sup_{|\phi|_{0.5}=1} |f\phi|_{0.5}.$$

It is clear that

$$|f\phi|_0 \leq |f|_{L^\infty(\Gamma)}\,|\phi|_0 \tag{4.8}$$

and

$$|f\phi|_1 = |f\phi|_0 + |\bar{\partial} f\, \phi|_0 + |f\, \bar{\partial}\phi|_0.$$

By the embeddings

$$H^{0.25}(\Gamma) \subset L^{8/3}(\Gamma), \quad H^{0.75}(\Gamma) \subset L^8(\Gamma), \quad H^{1.25}(\Gamma) \subset L^\infty(\Gamma),$$

we see that

$$|f\phi|_1 \leq C|f|_{1.25}|\phi|_1. \tag{4.9}$$

Using interpolation between the inequalities (4.8) and (4.9) shows that

$$|f\phi|_{0.5} \leq C|f|_{1.25}|\phi|_{0.5},$$

which in turn proves the claim.

Thus, using the inequality (4.7), we find that

$$|(v^+ - v^-) \cdot n_{tt}|^2_{-0.5} \leq C|v^+ - v^-|^2_{1.25}\left[|\bar{\partial} v_t^+|^2_{-0.5} + |v^+|^2_{2.25}\right] \leq C \sup_{t\in[0,T]} \mathscr{E}(t).$$

Now we turn to the estimate of the last term $|(v_t^+ - v_t^-) \cdot n_t|^2_{-0.5}$. By Sobolev's embedding,

$$\begin{aligned} |v_t^+ \cdot n_t|^2_{-0.5} &\leq C|v_t^+ \cdot n_t|^2_0 \leq C\|v_t^+\|^2_{L^4(\Gamma)}\|\bar{\partial} v^+\|^2_{L^4(\Gamma)} \leq C\|v_t^+\|^2_{1,+}\|v^+\|^2_{2,+} \\ &\leq C\left[\|v_t^+\|^2_{1.5,+}\|v^+\|^2_{1.5,+} + \|v_t^+\|^2_{0,+}\|v^+\|^2_{3,+}\right] \leq C \sup_{t\in[0,T]} \mathscr{E}(t). \end{aligned}$$

The estimate for $v_t^- \cdot n_t$ is the same and thus $v_{tt}^- \cdot n$ shares the same $H^{-0.5}(\Gamma)$ bound as $v_{tt}^+ \cdot n$. Therefore

$$|v_{tt}^+ \cdot n|^2_{-0.5} + |v_{tt}^- \cdot n|^2_{-0.5} \leq C \sup_{t\in[0,T]} \mathscr{E}(t). \tag{4.10}$$

Step 3 (Estimates for q_t^-). Time-differentiating (1.3b) and taking the Lagrangian divergence of the resulting equation, we find that q_t^- satisfies

$$A_i^j(A_i^k q_{t,k}^-)_{,j} = -\epsilon\left[A_i^j v_{tt,j}^{-i} + A_i^j(\mathrm{w}^\ell v_{,\ell}^{-i})_{t,j}\right] - A_i^j[(A_i^k)_t q_{,k}^-]_{,j} \quad \text{in } \Omega^-, \tag{4.11a}$$

$$A_i^j A_i^k q_{t,k}^- N_j = -\left[\epsilon v_{tt}^{-i} + \epsilon(\mathrm{w}^\ell v_{,\ell}^{-i})_t + (A_i^k)_t q_{,k}^-\right] A_i^j N_j \quad \text{on } \Gamma, \tag{4.11b}$$

$$q_{t,k}^- N_k = -\epsilon(\mathrm{w}^\ell v_{,\ell}^{-i})_t N_i \quad \text{on } \partial\mathscr{D}. \tag{4.11c}$$

The goal is to estimate the $H^1(\Omega^-)$-norm of q_t^-. By elliptic regularity, it suffices to estimate the $H^1(\Omega)'$-norm of the interior forcing, and the $H^{-0.5}(\Gamma)$-norm of the boundary forcing.

Similar to Step 2, in order to find an ϵ-independent bound for $\|A_i^j v_{tt,j}^-\|^2_{H^1(\Omega^-)'}$, we need to estimate $\|(A_i^j)_t v_{t,j}^{-i}\|_{H^1(\Omega^-)'}$. To be more specific, we need to obtain an ϵ-independent bound for $|(\sqrt{g}n)_t \cdot v_t^-|_{-0.5}$. It suffices to estimate $|v_t^- \cdot n|_{-0.5}$ and $|v_t^- \cdot n_t|_{-0.5}$. It is clear that

$$|v_t^- \cdot n|^2_{-0.5} \leq C \sup_{t\in[0,T]} \mathscr{E}(t)$$

and the estimate for $v_t^- \cdot n_t$ is the same as the one for $v_t^+ \cdot n_t$ previously used to establish (4.10); therefore,

$$\|A_i^j v_{tt,j}^{-i}\|^2_{H^1(\Omega^-)'} \leq C \sup_{t\in[0,T]} \mathscr{E}(t).$$

We now consider the $H^1(\Omega^-)'$-norm of $A_i^j(\mathrm{w}^\ell v_{,\ell}^{-i})_{t,j}$. Again by (1.3c),

$$\begin{aligned} A_i^j(\mathrm{w}^\ell v_{,\ell}^{-i})_{t,j} = {} & A_i^j \mathrm{w}_{t,j}^\ell v_{,\ell}^{-i} + A_i^j \mathrm{w}_t^\ell v_{,\ell j}^{-i} + A_i^j \mathrm{w}_{,j}^\ell v_{t,\ell}^{-i} \\ & - (A_i^j)_t \mathrm{w}^\ell v_{,j\ell}^{-i} - (A_i^j)_{,\ell} \mathrm{w}^\ell v_{t,j}^{-i} - (A_i^j)_{t,\ell} \mathrm{w}^\ell v_{,j}^{-i}. \end{aligned} \tag{4.12}$$

Similar to (4.4),

$$\begin{aligned}\langle A_i^j \mathrm{w}^\ell_{t,j} v^{-i}_{,\ell}, f\rangle_{H^1(\Omega^-)} &= -\int_{\Omega^-} A_i^j \mathrm{w}^\ell_t (v^{-i}_{,\ell} f_{,j} + v^{-i}_{,\ell j} f) dx + \int_{\Gamma\cup\partial\mathscr{D}} \sqrt{g} n^i \mathrm{w}^\ell_t v^{-i}_{,\ell} f dS \\ &= \int_{\Omega^-} \mathrm{w}^\ell_t \big[(A_i^j)_{,\ell} v^{-i}_{,j} f - A_i^j v^{-i}_{,\ell} f_{,j}\big] dx + \int_{\Gamma\cup\partial\mathscr{D}} \sqrt{g} n^i \mathrm{w}^\ell_t v^{-i}_{,\ell} f dS,\end{aligned}$$

where $\mathrm{w}\cdot N = 0$ on Γ and $v^-\cdot N = 0$ on $\partial\mathscr{D}$ are used in the second equality to eliminate the boundary term due to the integration by parts with respect to D_{x_ℓ}; hence,

$$\begin{aligned}\|A_i^j \mathrm{w}^\ell_{t,j} v^{-i}_{,\ell}\|^2_{H^1(\Omega^-)'} &\le C\big[\|\mathrm{w}_t Dv^-\|^2_{0,+} + \|\mathrm{w}_t DADv^-\|^2_{L^{6/5}(\Omega^-)} + \|\mathrm{w}_t Dv^-\|^2_{L^{4/3}(\Gamma)}\big] \\ &\le C \sup_{t\in[0,T]} \mathscr{E}(t). \end{aligned} \tag{4.13}$$

Similarly, all the other terms in the right-hand side of (4.12) share the same $H^1(\Omega^-)'$ bound, so that

$$\|A_i^j (\mathrm{w}^\ell v^{-i}_{,\ell})_{t,j}\|^2_{H^1(\Omega^-)'} \le C \sup_{t\in[0,T]} \mathscr{E}(t).$$

Next, we estimate the $H^{-0.5}(\Gamma)$-norm of the boundary forcing. Because of (4.10), it suffices to estimate $|(\mathrm{w}^\ell v^{-i}_{,\ell})_t n^i|_{-0.5}$. However, because of (4.13), it suffices to estimate $\|(\mathrm{w}^\ell v^-_{,\ell})_t\|_{0,-}$ and it is easy to see that

$$\|(\mathrm{w}^\ell v^-_{,\ell})_t\|^2_{0,-} \le C\big[\|v^-_t\|^2_{1,-} + \|v^+_t\|^2_{1,+}\big] \le C \sup_{t\in[0,T]} \mathscr{E}(t).$$

Combining (4.2) and all the estimates above, we find that for all $t \in [0, T]$,

$$\frac{1}{\epsilon^2}\|q^-_t(t)\|^2_{1,-} \le C \sup_{t\in[0,T]} \mathscr{E}(t). \tag{4.14}$$

Step 4 (Estimates for q^+ and q^+_t). By studying the Neumann problems

$$\begin{aligned}A_i^j (A_i^k q^+_{,k})_{,j} &= \epsilon A_r^j v^{+r}_{,s} A_j^s v^{+i}_{,j} \quad \text{in } \Omega^+, \\ A_i^j A_i^k q^+_{,k} N_j &= -\epsilon v^{+i}_t A_i^j N_j \quad \text{on } \Gamma\end{aligned}$$

and

$$\begin{aligned}A_i^j (A_i^k q^+_{t,k})_{,j} &= -\epsilon A_i^j v^{+i}_{tt,j} + A_i^j [A_r^k v^{+r}_{,s} A_i^s q^+_{,k}]_{,j} \\ &= \epsilon\big[2A_r^j v^{+r}_{,s} A_i^s v^{+i}_{t,j} + (A_t^j)_{tt} v^{+i}_{,j}\big] + A_i^j [A_r^k v^{+r}_{,s} A_i^s q^+_{,k}]_{,j} \quad \text{in } \Omega^+, \\ A_i^j A_i^k q^+_{t,k} N_j &= -\epsilon v^{+i}_{tt} A_i^j N_j - A_i^j A_i^k q^+_{,k} N_j \quad \text{on } \Gamma,\end{aligned}$$

we obtain that for $t \in [0, T]$,

$$\|q^+(t)\|^2_{2.5,+} + \|q^+_t(t)\|^2_{1,+} \le C \sup_{t\in[0,T]} \mathscr{E}(t). \tag{4.15}$$

These 4 steps conclude the proposition. □

4.2. *Uniform Boundedness of* $\|v_t^-\|_{0,-}$

Having $\|q_t^-\|_{1,-}^2$ uniformly bounded in ϵ, we can prove the last part of the basic assumptions of Section 3.2. By (1.3b), (3.10), (3.9) and (4.1) we find that

$$\begin{aligned}\|v_t^-\|_{0,-}^2 &\le C\|\mathrm{w}\cdot Dv^-\|_{0,-}^2 + \frac{C}{\epsilon^2}\|q^-\|_{1,-}^2\\ &\le C\|\mathrm{w}\|_{1,-}^2\|Dv^-\|_{0.5,-}^2 + \frac{C}{\epsilon^2}\Big[\|q_0^-\|_{1,-}^2 + \int_0^t \|q_t^-\|_{1,-}^2\, ds\Big]\\ &\le C + CT\sup_{t\in[0,T]}\mathscr{E}(t).\end{aligned}$$

Therefore, (3.10), (3.9) and (4.1) imply that $\|v_t^-\|_{0,-}^2 \le C + CT\sup_{t\in[0,T]}\mathscr{E}(t)$ and thus by choosing T small enough, the basic assumptions imply

$$\|D\psi\|_{2,\pm}^2 + \|v\|_{1.5,\pm}^2 + \|v_t\|_{0,\pm}^2 + \|\mathrm{w}\|_{1.5,-}^2 + \|\mathrm{w}_t\|_{0,-}^2 \le C, \tag{4.16}$$

where C is the generic constant defined in Section 3.2.

4.3. *Estimates for* $v_{tt}^+\cdot\bar\partial\psi$ *and* $v_{tt}^-\cdot\bar\partial\psi$

Proposition 4.2. *Let* $v^\pm$ *be the solution to* (1.3). *Then*

$$|v_{tt}^+\cdot\bar\partial\psi|_{-0.5}^2 + |v_{tt}^-\cdot\bar\partial\psi|_{-0.5}^2 \le C\sup_{t\in[0,T]}\mathscr{E}(t). \tag{4.17}$$

Proof. First, note that since $D\psi = A^{-1}$, we have

$$f_{,\gamma} = \psi^i_{,\gamma}A^j_i f_{,j}. \tag{4.18}$$

It follows from (1.3a), (1.3b) and (4.18) that

$$v_{tt}^+\cdot\psi_{,\delta} = -\psi^i_{,\delta}A^j_i q^+_{t,j} - \psi^i_{,\delta}(A^j_i)_t q^+_{,j} = -q^+_{t,\delta} + A^j_i v^{+i}_{,\delta} q^+_{,j}, \tag{4.19a}$$

$$v_{tt}^-\cdot\psi_{,\delta} = -\frac{1}{\epsilon}q^-_{t,\delta} + \frac{1}{\epsilon}A^j_i v^{+i}_{,\delta}q^-_{,j} - \mathrm{w}^\ell_t v^{-i}_{,\ell}\psi^i_{,\delta} - \mathrm{w}^\ell v^{-i}_{t,\ell}\psi^i_{,\delta}. \tag{4.19b}$$

As a consequence,

$$\begin{aligned}&|v_{tt}^+\cdot\psi_{,\delta}|_{-0.5}^2 + |v_{tt}^-\cdot\psi_{,\delta}|_{-0.5}^2\\ &\quad\le \frac{1}{\epsilon^2}|q^-_{t,\delta}|_{-0.5}^2 + |q^+_{t,\delta}|_{-0.5}^2 + |A|_{L^\infty(\Gamma)}^2|\bar\partial v^+|_{L^4(\Gamma)}^2\left[\frac{1}{\epsilon^2}\|Dq^-\|_{L^4(\Gamma)}^2 + \|Dq^+\|_{L^4(\Gamma)}^2\right]\\ &\qquad + |\mathrm{w}_t|_{L^4(\Gamma)}^2|Dv^-|_{L^4(\Gamma)}^2|\bar\partial\psi|_{L^\infty(\Gamma)}^2 + |\mathrm{w}^\ell v^-_{t,\ell}\cdot\bar\partial\psi|_{-0.5}^2\\ &\quad\le C\sup_{t\in[0,T]}\mathscr{E}(t) + C\|v^+\|_{2,+}^2\left[\frac{1}{\epsilon^2}\|q^-\|_{2,-}^2 + \|q^+\|_{2,+}^2\right]\\ &\qquad + C\|\mathrm{w}_t\|_{1,-}^2\|v^-\|_{2,-}^2 + C|\mathrm{w}^\ell v^-_{t,\ell}|_{-0.5}^2.\end{aligned}$$

By interpolation and Young's inequality,

$$\begin{aligned}&\|v^+\|_{2,+}^2\left[\frac{1}{\epsilon^2}\|q^-\|_{2,-}^2+\|q^+\|_{2,+}^2\right]\\&\quad\le C\|v^+\|_{1.5,+}^{4/3}\|v^+\|_{3,+}^{2/3}\\&\qquad\times\left[\frac{1}{\epsilon^2}\|q^-\|_{2.5,-}^{4/3}\|q^-\|_{1,-}^{2/3}+\|q^+\|_{2.5,+}^{4/3}\|q^+\|_{1,+}^{2/3}\right]\le C\sup_{t\in[0,T]}\mathscr{E}(t),\end{aligned}$$

and the same upper bound holds for $\|\mathrm{w}_t\|_{1,-}^2\|v^-\|_{2,-}^2$. For the last term, since $\mathrm{w}\cdot N=0$ on Γ and with the identity $\mathrm{w}=(\mathrm{w}\cdot N)N+g_0^{\alpha\beta}(\mathrm{w}\cdot e_{,\alpha})e_{,\beta}$, we find that

$$\mathrm{w}^\ell v_{t,\ell}^-=(\mathrm{w}\cdot N)N^\ell v_{t,\ell}^-+g_0^{\alpha\beta}(\mathrm{w}\cdot e_{,\alpha})v_{t,\ell}^-e_{,\beta}^\ell=g_0^{\alpha\beta}(\mathrm{w}\cdot e_{,\alpha})v_{t,\beta}^-;$$

hence

$$|\mathrm{w}^\ell v_{t,\ell}^-|_{-0.5}^2\le C|\bar\partial v_t^-|_{-0.5}^2\le C\sup_{t\in[0,T]}\mathscr{E}(t).$$

(4.17) then follows from summing all the estimates above. □

4.4. *The ϵ-Independent Energy Estimates*

It remains to establish the estimates for $\bar\partial v\cdot N$, $\bar\partial v_t\cdot N$ and $\bar\partial\eta^+\cdot N$.

Theorem 4.1. *The solution $v^\pm$ to* (1.3) *satisfies the following estimate*:

$$\begin{aligned}&\sup_{t\in[0,T]}\left[\|v_{tt}^+(t)\|_{0,+}^2+\epsilon\|v_{tt}^-(t)\|_{0,-}^2+|\bar\partial v_t(t)\cdot N|_{0,\pm}^2+|\bar\partial v(t)\cdot N|_{1.5,\pm}^2+|\bar\partial\eta^+(t)\cdot N|_3^2\right]\\&\quad\le C_\delta\mathscr{M}_0+\delta\sup_{t\in[0,T]}\mathscr{E}(t)+C_\delta T\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}(t)\right).\end{aligned}\tag{4.20}$$

Proof. We first derive the estimate

$$\begin{aligned}&\sup_{t\in[0,T]}\left[\|v_{tt}^+(t)\|_{0,+}^2+\epsilon\|v_{tt}^-(t)\|_{0,-}^2+|\bar\partial v_t(t)\cdot N|_{0,\pm}^2\right]\\&\quad\le C_\delta\mathscr{M}_0+\delta\sup_{t\in[0,T]}\mathscr{E}(t)+C_\delta T\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}(t)\right).\end{aligned}$$

Twice time-differentiating (1.3a) and (1.3b) and testing the resulting equations against v_{tt}^+ and v_{tt}^-, respectively, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|v_{tt}^+\|_{0,+}^2+\epsilon\|v_{tt}^-\|_{0,-}^2\right]+\epsilon\int_{\Omega^-}\left[\mathrm{w}_{tt}^jv_{,j}^{-i}+2\mathrm{w}_t^jv_{t,j}^{-i}+\mathrm{w}^jv_{tt,j}^{-i}\right]v_{tt}^{-i}dx\\&\quad+\int_{\Omega^+}(A_i^jq_{,j}^+)_{tt}v_{tt}^{+i}dx+\int_{\Omega^-}(A_i^jq_{,j}^-)_{tt}v_{tt}^{-i}dx=0.\end{aligned}\tag{4.21}$$

First, note that $v^{-i}_{tt,j} v^{-i}_{tt} = \frac{1}{2}(|v_{tt}|^2)_{,j}$, so that

$$\int_{\Omega^-} \mathrm{w}^j v^{-i}_{tt,j} v^{-i}_{tt} dx = -\frac{1}{2} \int_{\Omega^-} \operatorname{div} \mathrm{w} |v_{tt}|^2 dx.$$

Second, using (4.3), we obtain that

$$\begin{aligned} \int_{\Omega^+} (A^j_i q^+_{,j})_{tt} v^{+i}_{tt} dx &= -\int_{\Omega^+} (A^j_i q^+)_{tt} v^{+i}_{tt,j} dx + \int_\Gamma (A^j_i q^+)_{tt} v^{+i}_{tt} N_j dS \\ &= -\int_{\Omega^+} \left[(A^j_i)_{tt} q^+ + 2(A^j_i)_t q^+_t \right] v^{+i}_{tt,j} dx \\ &\quad + \int_{\Omega^+} q^+_{tt} \left[(A^j_i)_{tt} v^{+i}_{,j} + 2(A^j_i)_t v^{+i}_{t,j} \right] dx + \int_\Gamma (A^j_i q^+ N_j)_{tt} v^{+i}_{tt} dS \end{aligned}$$

with a similar identity for the term $\int_{\Omega^-} (A^j_i q^-_{,j})_{tt} v^{-i}_{tt} dx$. Using these identities, (4.21) implies that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|v^+_{tt}\|^2_{0,+} + \epsilon \|v^-_{tt}\|^2_{0,-} \right] - \frac{\epsilon}{2} \int_{\Omega^-} \operatorname{div} \mathrm{w} |v^-_{tt}|^2 dx + \epsilon \int_{\Omega^-} \left[\mathrm{w}^j_{tt} v^{-i}_{,j} + \mathrm{w}^j_t v^{-i}_{t,j} \right] v^{-i}_{tt} dx \\ &\left.\begin{aligned} &- \int_{\Omega^+} \left[(A^j_i)_{tt} q^+ + 2(A^j_i)_t q^+_t \right] v^{+i}_{tt,j} dx - \int_{\Omega^-} \left[(A^j_i)_{tt} q^- + 2(A^j_i)_t q^-_t \right] v^{-i}_{tt,j} dx \\ &+ \int_{\Omega^+} \left[(A^j_i)_{tt} v^{+i}_{,j} + 2(A^j_i)_t v^{+i}_{t,j} \right] q^+_{tt} dx + \int_{\Omega^-} \left[(A^j_i)_{tt} v^{-i}_{,j} + 2(A^j_i)_t v^{-i}_{t,j} \right] q^-_{tt} dx \end{aligned}\right\} (\equiv \mathrm{I}) \\ &+ \underbrace{\int_\Gamma [A^j_i N_j (q^+ - q^-)]_{tt} v^{+i}_{tt} dS}_{\mathrm{O}} + \underbrace{\int_\Gamma (A^j_i q^- N_j)_{tt} (v^{+i}_{tt} - v^{-i}_{tt}) dS}_{\mathrm{J}} = 0. \end{aligned}$$

Henceforth, $\mathscr{R}$ denotes *lower-order remainder terms* that can be easily shown to verify the estimate

$$\left| \int_0^t \mathscr{R} ds \right| \leq \mathscr{M}_0 + CT\mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t) \right).$$

With this notation, the equality above is rewritten as

$$\frac{1}{2}\frac{d}{dt}\left[\|v^+_{tt}\|^2_{0,+} + \epsilon \|v^-_{tt}\|^2_{0,-} \right] + \mathrm{O} + \mathrm{I} + \mathrm{J} + \mathscr{R} = 0.$$

Step 1 (Estimates for the Surface Tension Term O). By the boundary condition (1.3d),

$$\int_\Gamma [A^j_i N_j (q^+ - q^-)]_{tt} v^{+i}_{tt} dS = \int_\Gamma \left[\sqrt{g} H n^j \right]_{tt} v^{+j}_{tt} dS.$$

Since the metric g and H are computed from η^+ whose time derivatives only involve $\partial^k_t v^+$ (which is bounded by $\mathscr{E}(t)$), exactly following (12.6) in [6] we find that

$$\begin{aligned} &\frac{1}{2} \int_\Gamma \sqrt{g} g^{\alpha\beta} (v^+_{t,\alpha} \cdot n)(v^+_{t,\beta} \cdot n) dS \\ &\quad \leq \int_0^t \int_\Gamma [A^j_i N_j (q^+ - q^-)]_{tt} v^{+i}_{tt} dS ds + \mathscr{M}_0 + CT\mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t) \right). \end{aligned} \tag{4.22}$$

Step 2 (The Estimates for Error Term I). Let $\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4$, the summands representing the four integrals contained in I:

$$\mathrm{I}_1 = -\int_{\Omega^+} \big[(A_i^j)_{tt} q^+ + 2(A_i^j)_t q_t^+\big] v_{tt,j}^{+i} dx, \quad \mathrm{I}_2 = -\int_{\Omega^-} \big[(A_i^j)_{tt} q^- + 2(A_i^j)_t q_t^-\big] v_{tt,j}^{-i} dx,$$

$$\mathrm{I}_3 = \int_{\Omega^+} \big[(A_i^j)_{tt} v_{,j}^{+i} + 2(A_i^j)_t v_{t,j}^{+i}\big] q_{tt}^+ dx, \quad \mathrm{I}_4 = \int_{\Omega^-} \big[(A_i^j)_{tt} v_{,j}^{-i} + 2(A_i^j)_t v_{t,j}^{-i}\big] q_{tt}^- dx.$$

Integrating by parts in D_{x_j} and using the fact that $(A_i^j)_{,j} = 0$ removing the higher order term $[(A_i^j)_{tt}]_{,j}$ which otherwise would have been problematic for our framework, it is easy to see that

$$\left| \int_0^t (\mathrm{I}_1 + \mathrm{I}_2) ds \right| \leq CT\mathscr{P}\bigg(\sup_{t\in[0,T]} \mathscr{E}(t) \bigg).$$

Next, we estimate I_4; the estimate for I_3 will follow in the same fashion. Integrating by parts in time, the most problematic terms to estimate are denoted by

$$\mathrm{I}_{41} = \int_0^t \int_{\Omega^-} A_r^j \psi_{ttt,s}^r A_i^s v_{,j}^{-i} q_t^- \, dx\, ds \quad \text{and} \quad \mathrm{I}_{42} = \int_0^t \int_{\Omega^-} (A_i^j)_t v_{tt,j}^{-i} q_t^- \, dx\, ds.$$

The same as the estimates for I_1 and I_2, integrating by parts with respect to D_{x_s} for I_{41} and D_{x_j} for I_{42}, we find that

$$\left| \int_0^t (\mathrm{I}_{41} + \mathrm{I}_{42}) ds \right| \leq CT\mathscr{P}\bigg(\sup_{t\in[0,T]} \mathscr{E}(t) \bigg)$$

so that

$$\left| \int_0^t \mathrm{I}\, ds \right| \leq \mathscr{M}_0 + CT\mathscr{P}\bigg(\sup_{t\in[0,T]} \mathscr{E}(t) \bigg),$$

where $\mathscr{M}_0$ comes from the temporal boundary terms appearing when integrating by parts in time for I_3 and I_4.

Step 3 (The Estimates for the Error Term J). Since $A^T N = \sqrt{g} n$, we find that

$$\mathrm{J} = \underbrace{\int_\Gamma (\sqrt{g} q^-)_{tt} (v_{tt}^+ - v_{tt}^-) \cdot n dS}_{\mathrm{J}_1} + 2 \underbrace{\int_\Gamma (\sqrt{g} q^-)_t n_t^i \cdot (v_{tt}^{+i} - v_{tt}^{-i}) dS}_{\mathrm{J}_2}$$
$$+ \underbrace{\int_\Gamma \sqrt{g} q^- (v_{tt}^+ - v_{tt}^-) \cdot n_{tt} dS}_{\mathrm{J}_3}.$$

The Estimates for J_1: by the boundary condition (1.3e),

$$\begin{aligned}(v_{tt}^+ - v_{tt}^-) \cdot n &= (v^- - v^+) \cdot n_{tt} + 2(v_t^- - v_t^+) \cdot n_t \\ &= [(v^+ - v^-) \cdot \psi_{,\delta}] g^{\gamma\delta} (v_{t,\gamma}^+ \cdot n) + 2[(v_t^+ - v_t^-) \cdot \psi_{,\delta}] g^{\gamma\delta} (v_{,\gamma}^+ \cdot n) \\ &\quad + \mathscr{P}_i(\bar\partial \psi, \bar\partial v^+)(v^{+i} - v^{-i}).\end{aligned} \tag{4.23}$$

Time integrating J_1 and integrating by parts in time, we find that

$$\int_0^t J_1\,ds = \underbrace{\Big[\int_\Gamma \big(\sqrt{g}q^-\big)_t (v_{tt}^+ - v_{tt}^-)\cdot n dS\Big](t)}_{J_{10}} - \int_\Gamma \big(\sqrt{g}q^-\big)_t(0)(w_2^+ - w_2^-)\cdot N dS$$
$$- \int_0^t \int_\Gamma \big(\sqrt{g}q^-\big)_t g^{\gamma\delta}[(v^+ - v^-)\cdot \psi_{,\delta}](v_{tt,\gamma}^+ \cdot n) dS\,ds \quad (\equiv J_{11})$$
$$- 3\int_0^t \int_\Gamma \big(\sqrt{g}q^-\big)_t g^{\gamma\delta}[(v_t^+ - v_t^-)\cdot \psi_{,\delta}](v_{t,\gamma}^+ \cdot n) dS\,ds$$
$$- 2\int_0^t \int_\Gamma \big(\sqrt{g}q^-\big)_t g^{\gamma\delta}[(v_{tt}^+ - v_{tt}^-)\cdot \psi_{,\delta}](v_{,\gamma}^+ \cdot n) dS\,ds \quad (\equiv J_{12})$$
$$+ \int_0^t \int_\Gamma \big(\sqrt{g}q^-\big)_t \big[\mathscr{P}_{ij}(\bar\partial\psi, \bar\partial v^+)\bar\partial v_t^{+i} + \mathscr{P}_j(\bar\partial\psi, \bar\partial v^+)\big](v^{+j} - v^{-j}) dS\,ds$$
$$= J_{10} + J_{11} + J_{12} - \int_\Gamma \big(\sqrt{g}q^-\big)_t(0)(w_2^+ - w_2^-)\cdot N dS + \int_0^t \mathscr{R} ds.$$

By (4.17),

$$|J_{12}| \le CT\mathscr{P}\bigg(\sup_{t\in[0,T]} \mathscr{E}(t)\bigg),$$

so we only need to estimate J_{10} and J_{11}.

For the temporal boundary term J_{10}, using (4.23), the fundamental theorem of calculus, and interpolation, we find that

$$|J_{10}| \le C\|v_t^+\|_{1,+}\|q_t^-\|_{1,-} + CT\mathscr{P}\bigg(\sup_{t\in[0,T]} \mathscr{E}(t)\bigg)$$
$$\le C_\delta \|v_t^+\|_0^2 + \delta\big[\|v_t^+\|_{1.5}^2 + \|q_t^-\|_1^2\big] + CT\mathscr{P}\bigg(\sup_{t\in[0,T]} \mathscr{E}(t)\bigg)$$
$$\le C_\delta + CT\mathscr{P}\bigg(\sup_{t\in[0,T]} \mathscr{E}(t)\bigg) + \delta \sup_{t\in[0,T]} \mathscr{E}(t).$$

As for J_{11}, integrating by parts with respect to $\bar\partial_\gamma$,

$$J_{11} = \int_0^t \int_\Gamma \big(\sqrt{g}q^-\big)_{t,\gamma} g^{\gamma\delta}[(v^+ - v^-)\cdot \psi_{,\delta}](v_{tt}^+ \cdot n) dS\,ds \quad (\equiv J_{111})$$
$$\left.\begin{aligned} &+ \int_0^t \int_\Gamma \big(\sqrt{g}q^-\big)_t \big[\mathscr{P}_i(\bar\partial^2\psi)(v^{+i} - v^{-i}) + \mathscr{P}(\bar\partial\psi, \bar\partial v)\big](v_{tt}^+ \cdot n) dS\,ds \\ &+ \int_0^t \int_\Gamma \big(\sqrt{g}q^-\big)_t \mathscr{P}_{ij}^\alpha(\bar\partial\psi)\bar\partial^2\psi^i(v^{+j} - v^{-j})(v_{tt}^+ \cdot \psi_{,\alpha}) dS\,ds \end{aligned}\right\} \equiv J_{112}.$$

By $H^{0.5}(\Gamma)$–$H^{-0.5}(\Gamma)$ duality, we find that J_{112} is bounded by $CT\mathscr{P}(\sup_{t\in[0,T]} \mathscr{E}(t))$.

For the term J_{111}, we add and subtract q^+ to obtain

$$J_{111} = \int_0^t \int_\Gamma [\sqrt{g}(q^- - q^+)]_{t,\gamma} g^{\gamma\delta}[(v^+ - v^-)\cdot \psi_{,\delta}](v_{tt}^+ \cdot n) dS\,ds$$
$$+ \int_0^t \int_\Gamma (\sqrt{g}q^+)_{t,\gamma} g^{\gamma\delta}[(v^+ - v^-)\cdot \psi_{,\delta}](v_{tt}^+ \cdot n) dS\,ds$$

$$= \int_0^t \int_\Gamma \left[(\sqrt{g}g^{\alpha\beta}\psi_{,\alpha})_{,\beta} \cdot n\right]_{t,\gamma} g^{\gamma\delta}[(v^+ - v^-) \cdot \psi_{,\delta}](v_{tt}^+ \cdot n)dS\,ds \quad (\equiv \mathrm{J}_{1111})$$
$$+ \int_0^t \int_\Gamma (\sqrt{g}q^+)_{t,\gamma} g^{\gamma\delta}[(v^+ - v^-) \cdot \psi_{,\delta}](v_{tt}^+ \cdot n)dS\,ds \quad (\equiv \mathrm{J}_{1112}).$$

In order to study the term J_{1111}, we integrate by parts with respect to $\bar{\partial}_\gamma$ and then in time (to move one time derivative from $v_{tt}^+ \cdot n$) and find that the most challenging term to estimate is

$$\int_0^t \int_\Gamma \left[(\sqrt{g}g^{\alpha\beta}\psi_{,\alpha})_{,\beta} \cdot n\right]_{tt} g^{\gamma\delta}[(v^+ - v^-) \cdot \psi_{,\delta}](v_t^+ \cdot n)_{,\gamma} dS\,ds.$$

Since

$$\begin{aligned}\left[(\sqrt{g}g^{\alpha\beta}\psi_{,\alpha})_{,\beta} \cdot n\right]_{tt} &= \left(\sqrt{g}g^{\alpha\beta}v_{t,\alpha}^+\right)_{,\beta} \cdot n + \sqrt{g}g^{\alpha\beta}g^{\gamma\delta}(v_{t,\gamma}^+ \cdot \psi_{,\delta})(\psi_{,\alpha\beta} \cdot n) \\ &\quad - \sqrt{g}\left[g^{\alpha\gamma}g^{\beta\delta}(v_{t,\gamma}^+ \cdot \psi_{,\delta} + v_{t,\delta}^+ \cdot \psi_{,\gamma}) + g^{\alpha\beta}g^{\gamma\delta}(v_{,\gamma}^+ \cdot n)(v_{,\delta}^+ \cdot n)\right] \\ &\quad \times (\psi_{,\alpha\beta} \cdot n) + \mathscr{P}_i^1(\bar{\partial}\psi, \bar{\partial}v^+)\bar{\partial}^2\psi^i + \mathscr{P}_i^2(\bar{\partial}\psi, \bar{\partial}v^+)\bar{\partial}^2 v^{+i},\end{aligned}$$

the most difficult term to estimate after integrating by parts with respect to $\bar{\partial}_\beta$ is

$$\int_0^t \int_\Gamma \sqrt{g}g^{\alpha\beta}g^{\gamma\delta}[(v^+ - v^-) \cdot \psi_{,\delta}](v_{t,\alpha}^+ \cdot n)(v_{t,\beta}^+ \cdot n)_{,\gamma} dS\,ds.$$

Now by

$$g^{\alpha\beta}(v_{t,\alpha}^+ \cdot n)(v_{t,\beta}^+ \cdot n)_{,\gamma} = \frac{1}{2}\left[g^{\alpha\beta}(v_{t,\alpha}^+ \cdot n)(v_{t,\beta}^+ \cdot n)\right]_{,\gamma} - \frac{1}{2}(g^{\alpha\beta})_{,\gamma}(v_{t,\alpha}^+ \cdot n)(v_{t,\beta}^+ \cdot n),$$

integrating by parts implies that the above integral is bounded by $CT\mathscr{P}(\sup_{t\in[0,T]} \mathscr{E}(t))$. Therefore,

$$|\mathrm{J}_{1111}| \le \mathscr{M}_0 + CT\mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right) + \left|\underbrace{\int_\Gamma (\sqrt{g}H)_t g^{\gamma\delta}[(v^+ - v^-) \cdot \psi_{,\delta}](v_t^+ \cdot n)_{,\gamma} dS}_{\mathrm{K}_1}\right|,$$

where $\mathscr{M}_0$ and the term K_1 arises from the temporal boundary term when integrating by parts in time for J_{1111}. However, similar to the estimate of J_{10}, the temporal boundary term K_1 can be estimated as

$$\begin{aligned}|\mathrm{K}_1| &\le C\|v^+\|_{2.5,+}\|v_t^+\|_{1.5,+} + CT\mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right) \\ &\le C_\delta\|v^+\|_{0,+}^2 + \delta\left[\|v^+\|_{3,+}^2 + \|v_t^+\|_{1.5,+}^2\right] + CT\mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right) \\ &\le C_\delta + \delta \sup_{t\in[0,T]} \mathscr{E}(t) + CT\mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right).\end{aligned}$$

As for J_{1112}, by the divergence theorem, since $A_i^j v_{tt,j}^{+i}$ is a lower order term thanks to incompressibility, we find that

$$\mathrm{J}_{1112} = \underbrace{\int_0^t \int_{\mathscr{U}} \chi \frac{1}{\sqrt{g}} (\sqrt{g} q^+)_{,t,\gamma j} g^{\gamma\delta} [(v^+ - v^-) \cdot \psi_{,\delta}] A_i^j v_{tt}^{+i} dx\, ds}_{\mathrm{K}_2}$$

$$+ \int_0^t \int_{\mathscr{U}} \chi \frac{1}{\sqrt{g}} (\sqrt{g} q^+)_{,t,\gamma} g^{\gamma\delta} [(v^+ - v^-) \cdot \psi_{,\delta}] A_i^j v_{tt,j}^{+i} dx\, ds$$

$$+ \int_0^t \int_{\mathscr{U}} (\sqrt{g} q^+)_{,t,\gamma} \Big[\chi \frac{1}{\sqrt{g}} g^{\gamma\delta} [(v^+ - v^-) \cdot \psi_{,\delta}] \Big]_{,j} A_i^j v_{tt}^{+i} dx\, ds = \mathrm{K}_2 + \int_0^t \mathscr{R} ds,$$

where χ is a smooth cut-off function supported around Γ with $\chi = 1$ on Γ, and with the support of χ taken sufficiently close to Γ so that the tangential derivatives are well-defined. We set

$$\mathscr{U} = \mathrm{supp}(\chi) \cap \Omega^+.$$

To estimate K_2, the structure of the Euler equations has to be used. In fact,

$$\Bigg| \mathrm{K}_2 + \underbrace{\int_0^t \int_{\mathscr{U}} \chi v_{tt,\gamma}^{+i} g^{\gamma\delta} [(v^+ - v^-) \cdot \psi_{,\delta}] v_{tt}^{+i} dx\, ds}_{\mathrm{K}_3} \Bigg| \le CT\mathscr{P}\Big(\sup_{t\in[0,T]} \mathscr{E}(t) \Big),$$

where we use the identity

$$-\big(\sqrt{g} v_t^{+i}\big)_{t,\gamma} = \big(\sqrt{g} A_i^j q_{,j}^+\big)_{t,\gamma} = A_i^j \big(\sqrt{g} q_{,j}^+\big)_{t,\gamma} - (A_i^j)_t \big(\sqrt{g} q_{,j}^+\big)_{,\gamma} - (A_i^j)_{,\gamma} \big(\sqrt{g} q_{,j}^+\big)_t$$

to replace $\frac{1}{\sqrt{g}} A_i^j (\sqrt{g} q^+)_{,t,\gamma j}$ in K_2 by $v_{tt,\gamma}^{+i}$ in K_3.

It then suffices to estimate K_3 to complete the estimate of J_1. However, since $2 v_{tt,\gamma}^{+i} v_{tt}^{+i} = (|v_{tt}^+|^2)_{,\gamma}$, integrating by parts with respect to $\bar{\partial}_\gamma$ implies that K_3 is bounded by $CT\mathscr{P}(\sup_{t\in[0,T]} \mathscr{E}(t))$.

Therefore, combining all the estimates above we find that

$$\left| \int_0^t \mathrm{J}_1 ds \right| \le (C_\delta + \mathscr{M}_0) + CT\mathscr{P}\Big(\sup_{t\in[0,T]} \mathscr{E}(t) \Big) + \delta \sup_{t\in[0,T]} \mathscr{E}(t). \tag{4.24}$$

The Estimates for $\mathbf{J}_2$: we first note that by (4.19a,b),

$$(v_{tt}^+ - v_{tt}^-) \cdot \psi_{,\delta} = \frac{1}{\epsilon} q_{t,\delta}^- - q_{t,\delta}^+ + A_i^j v_{,\delta}^{+i} \left(\frac{1}{\epsilon} q_{,j}^+ - q_{,j}^- \right) + \mathrm{w}_t^\ell v_{,\ell}^{-i} \psi_{,\delta}^i + \mathrm{w}^\ell v_{t,\ell}^{-i} \psi_{,\delta}^i. \tag{4.25}$$

Therefore, by $n_t = -g^{\gamma\delta} (v_{,\gamma}^+ \cdot n) \psi_{,\delta}$ and (4.25), we find that

$$\mathrm{J}_2 = -\int_\Gamma \big(\sqrt{g} q^-\big)_t g^{\gamma\delta} (v_{,\gamma}^+ \cdot n) \left[\frac{1}{\epsilon} q_{t,\delta}^- - q_{t,\delta}^+ + A_i^j v_{,\delta}^{+i} \left(\frac{1}{\epsilon} q_{,j}^+ - q_{,j}^- \right) \right] dS$$

$$- \int_\Gamma \big(\sqrt{g} q^-\big)_t g^{\gamma\delta} (v_{,\gamma}^+ \cdot n) \big[\mathrm{w}_t^\ell v_{,\ell}^{-i} \psi_{,\delta}^i + \mathrm{w}^\ell v_{t,\ell}^{-i} \psi_{,\delta}^i \big] dS.$$

It is easy to see that the second integral is bounded by $C\epsilon\mathscr{P}(\sup_{t\in[0,T]}\mathscr{E}(t))$ (the presence of ϵ is due to the estimate of $\left(\sqrt{g}q^-\right)_t$). The most problematic term of the first integral is when the time derivative acts on q and in this case, by $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality, (4.2), (4.14) and (4.15) imply that

$$\left|\int_\Gamma \left(\sqrt{g}q^-\right)_t g^{\gamma\delta}(v^+_{,\gamma}\cdot n)\left[\frac{1}{\epsilon}q^-_{t,\delta} - q^+_{t,\delta} + A^j_i v^{+i}_{,\delta}\left(\frac{1}{\epsilon}q^+_{,j} - q^-_{,j}\right)\right]dS\right| \le C\epsilon\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}(t)\right).$$

Therefore,

$$\int_0^t \mathrm{J}_2 ds \le C\epsilon T\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}(t)\right). \tag{4.26}$$

The Estimates for J_3: by (4.25) we find that

$$\begin{aligned}\mathrm{J}_3 &= -\int_\Gamma q^-\sqrt{g}g^{\gamma\delta}(v^+_{t,\gamma}\cdot n)[(v^+_{tt}-v^-_{tt})\cdot\psi_{,\delta}]dS + \int_\Gamma \mathscr{P}_i(\bar\partial\psi,\bar\partial v^+)q^-(v^{+i}_{tt}-v^{-i}_{tt})dS\\ &= -\underbrace{\int_\Gamma \sqrt{g}g^{\gamma\delta}q^-(v^+_{t,\gamma}\cdot n)\left(\frac{1}{\epsilon}q^-_{t,\delta}-q^+_{t,\delta}\right)dS}_{\mathrm{K}_4} - \int_\Gamma \sqrt{g}g^{\gamma\delta}q^-(v^+_{t,\gamma}\cdot n)A^j_i v^{+i}_{,\delta}\left(\frac{1}{\epsilon}q^+_{t,j}-q^-_{t,j}\right)dS\\ &\quad - \int_\Gamma \sqrt{g}g^{\gamma\delta}q^-(v^+_{t,\gamma}\cdot n)[\mathrm{w}^\ell_t v^{-i}_{,\ell}\psi^i_{,\delta} + \mathrm{w}^\ell v^{-i}_{t,\ell}\psi^i_{,\delta}]dS + \int_\Gamma \mathscr{P}_i(\bar\partial\psi,\bar\partial v^+)q^-(v^{+i}_{tt}-v^{-i}_{tt})dS\\ &= -\mathrm{K}_4 + \mathscr{R}.\end{aligned}$$

Before estimating K_4, we first note that by the divergence theorem,

$$\begin{aligned}\int_\Gamma \sqrt{g}g^{\gamma\delta}q^-(v^+_{t,\gamma}\cdot n)q^+_{t,\delta}dS &= \int_\Gamma q^- g^{\gamma\delta}(v^{+i}_{t,\gamma}A^j_i N_j)q^+_{t,\delta}dS\\ &= \int_{\mathscr{U}} \chi q^- g^{\gamma\delta}A^j_i v^{+i}_{t,\gamma}q^+_{t,\delta j}dx + \int_{\mathscr{U}} \chi q^- g^{\gamma\delta}A^j_i v^{+i}_{t,\gamma j}q^+_{t,\delta}dx\\ &\quad + \int_{\mathscr{U}} (\chi q^- g^{\gamma\delta})_{,j}A^j_i v^{+i}_{t,\gamma}q^+_{t,\delta}dx.\end{aligned}$$

We then follow the estimate of J_{1112} and obtain the inequality

$$\left|\int_{\mathscr{U}} \chi q^- g^{\gamma\delta}A^j_i v^{+i}_{t,\gamma}q^+_{t,\delta j}dx + \int_{\mathscr{U}} \chi q^- g^{\gamma\delta}v^{+i}_{t,\gamma}v^{+i}_{tt,\delta}dx\right| \le C\epsilon\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}(t)\right).$$

By interpolating $v^+_t \in H^1(\Omega^+)$ between $L^2(\Omega^+)$ and $H^{1.5}(\Omega^+)$,

$$\begin{aligned}\left|\int_{\mathscr{U}} \chi q^- g^{\gamma\delta}v^{+i}_{t,\gamma}v^{+i}_{t,\delta}dx\right| &\le C\|q^-\|_{L^6(\Omega^-)}\|v^+_t\|_{W^{1,3}(\Omega^+)}\|v^+_t\|_{1,+}\\ &\le C_\delta\epsilon\left[\mathscr{M}_0 + T\sup_{t\in[0,T]}\mathscr{E}(t)\right]\|v^+_t\|^2_{1,+} + \epsilon\delta\|v^+_t\|^2_{1.5,+},\\ &\le C_\delta\epsilon\left[\mathscr{M}_0 + T\sup_{t\in[0,T]}\mathscr{E}(t)\right] + \epsilon\delta\sup_{t\in[0,T]}\mathscr{E}(t).\end{aligned} \tag{4.27}$$

Therefore,

$$\left|\int_0^t \int_\Gamma \sqrt{g} g^{\gamma\delta} q^-(v^+_{t,\gamma}\cdot n) q^+_{t,\delta} dS\, ds\right|$$
$$\leq \epsilon \mathscr{M}_0 + C\epsilon T \mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right) + \left|\left[\int_{\mathscr{U}} \chi q^- g^{\gamma\delta} v^{+i}_{t,\gamma} v^{+i}_{t,\delta} dx\right](t)\right|$$
$$\leq C_\delta \epsilon \mathscr{M}_0 + C_\delta \epsilon T \mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right) + \epsilon\delta \sup_{t\in[0,T]} \mathscr{E}(t). \tag{4.28}$$

It remains to estimate $\frac{1}{\varepsilon}\int_\Gamma \sqrt{g} g^{\gamma\delta} q^-(v^+_{t,\gamma}\cdot n) q^-_{t,\delta} dS$. By adding and subtracting q^+, using (1.3d) and (4.28) we find that

$$\frac{1}{\epsilon}\left|\int_0^t \int_\Gamma \sqrt{g} g^{\gamma\delta} q^-(v^+_{t,\gamma}\cdot n) q^-_{t,\delta} dS\, ds\right|$$
$$= \frac{1}{\epsilon}\left|\int_0^t \int_\Gamma \sqrt{g} g^{\gamma\delta} q^-(v^+_{t,\gamma}\cdot n)(q^- - q^+ + q^+)_{t,\delta} dS\, ds\right|$$
$$\leq C_\delta \mathscr{M}_0 + C_\delta T \mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right) + \delta \sup_{t\in[0,T]} \mathscr{E}(t)$$
$$+ \frac{1}{\epsilon}\left|\int_0^t \int_\Gamma \sqrt{g} g^{\gamma\delta} q^-(v^+_{t,\gamma}\cdot n)\left[\frac{1}{\sqrt{g}}(\sqrt{g} g^{\alpha\beta}\psi_{,\alpha})_{,\beta}\cdot n\right]_{t,\delta} dS\, ds\right|.$$

As for the last term, since $\psi_{,\alpha}\cdot n = 0$, the most problematic term appears when the time derivative hits $\psi_{,\alpha}$. Integrating by parts with respect to $\bar{\partial}_\beta$ for that term, we obtain that

$$\int_\Gamma \sqrt{g} g^{\gamma\delta} q^-(v^+_{t,\gamma}\cdot n)\left[\frac{1}{\sqrt{g}}(\sqrt{g} g^{\alpha\beta}\psi_{,\alpha})_{,\beta}\cdot n\right]_{t,\delta} dS$$
$$= -\int_\Gamma \sqrt{g} g^{\alpha\beta} g^{\gamma\delta} q^-(v^+_{t,\beta\gamma}\cdot n)(v^+_{,\alpha\delta}\cdot n) dS + \int_\Gamma \bar{\partial} q^-(\bar{\partial} v^+_t\cdot n)\mathscr{P}(\bar{\partial}\psi)\bar{\partial}^2 v^+ dS$$
$$+ \int_\Gamma q^-(\bar{\partial} v^+_t\cdot n)\mathscr{P}_i(\bar{\partial}\psi, \bar{\partial} v^+)\bar{\partial}^3\psi^i dS$$
$$+ \int_\Gamma q^-(\bar{\partial} v^+_t\cdot n)\left[\mathscr{P}^1_{ij}(\bar{\partial}\psi, \bar{\partial} v^+)\bar{\partial}^2\psi^j + \mathscr{P}^2_{ij}(\bar{\partial}\psi, \bar{\partial} v^+)\bar{\partial}^2 v^{+j}\right]\bar{\partial}^2\psi^i dS$$
$$= -\frac{1}{2}\frac{\partial}{\partial t}\int_\Gamma \sqrt{g} g^{\alpha\beta} g^{\gamma\delta} q^- v^+_{,\beta\gamma} v^+_{,\alpha\delta} + \mathscr{R}.$$

Consequently,

$$\frac{1}{\epsilon}\left|\int_0^t \int_\Gamma \sqrt{g} g^{\gamma\delta} q^-(v^+_{t,\gamma}\cdot n)\left[\frac{1}{\sqrt{g}}(\sqrt{g} g^{\alpha\beta}\psi_{,\alpha})_{,\beta}\cdot n\right]_{t,\delta} dS\, ds\right|$$
$$\leq C_\delta \mathscr{M}_0 + C_\delta T \mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right) + \delta \sup_{t\in[0,T]} \mathscr{E}(t)$$
$$+ \frac{C}{\epsilon}\left|\left[\int_\Gamma \sqrt{g} g^{\gamma\delta} q^-(v^+_{,\alpha\delta}\cdot n)(v^+_{,\beta\gamma}\cdot n) dS\right](t)\right|.$$

Similar to (4.27), we find that

$$\frac{1}{\epsilon}\left|\left[\int_\Gamma \sqrt{g}g^{\gamma\delta}q^-(v^+_{,\alpha\delta}\cdot n)(v^+_{,\beta\gamma}\cdot n)dS\right](t)\right| \\ \leq C_\delta\left[\mathscr{M}_0 + T\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}(t)\right)\right] + \delta\sup_{t\in[0,T]}\mathscr{E}(t).$$

The estimate of J_3 then follows from combining the above estimates:

$$\left|\int_0^t \text{J}_3 ds\right| \leq C_\delta\mathscr{M}_0 + C_\delta T\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}(t)\right) + \delta\sup_{t\in[0,T]}\mathscr{E}(t). \tag{4.29}$$

Combining (4.22), (4.24), (4.26) and (4.29), we find that

$$\sup_{t\in[0,T]}\left[\|v^+_{tt}(t)\|^2_{0,+} + \epsilon\|v^-_{tt}(t)\|^2_{0,-} + |\bar\partial v^+_t(t)\cdot n(t)|^2_0\right] \\ \leq C_\delta\mathscr{M}_0 + \delta\sup_{t\in[0,T]}\mathscr{E}(t) + C_\delta T\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}(t)\right).$$

It follows that

$$\sup_{t\in[0,T]}\left[\|v^+_{tt}(t)\|^2_{0,+} + \epsilon\|v^-_{tt}(t)\|^2_{0,-} + |\bar\partial v_t(t)\cdot N|^2_{0,\pm}\right] \\ \leq C_\delta\mathscr{M}_0 + \delta\sup_{t\in[0,T]}\mathscr{E}(t) + C_\delta T\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}(t)\right) \tag{4.30}$$

by the jump condition $v^+\cdot n = v^-\cdot n$ and the fact that $n(t) = N + \int_0^t n_t ds$.

Step 4 (Estimates for $|\bar\partial \boldsymbol{v}\cdot N|_{1.5,\pm}$ and $|\bar\partial\boldsymbol{\eta}^+\cdot N|_3$). Our goal is to establish an inequality of the type

$$\sup_{t\in[0,T]}\mathscr{E}(t) \leq C_\delta M_0 + \delta\sup_{t\in[0,T]}\mathscr{E}(t) + C_\delta\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}(t)\right)$$

with ϵ-independent C_δ. By (2.4), we only need (4.30)–(4.32) to ensure the ϵ-independence of C_δ. The real difficult part is estimate (4.30) which we prove in details. Following the proof of (12.33) and (12.34) in [6], by defining

$$\mathscr{E}_k(t) = |\bar\partial\psi\cdot N|^2_3 + \sum_{\ell=0}^k \|\partial_t^\ell v\|^2_{3-1.5\ell,\pm}$$

the proof in [6] implies

$$|\bar\partial v\cdot N|^2_{1.5,\pm} \leq C_\delta\mathscr{M}_0 + \delta\mathscr{E}_1(t) + C_\delta\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}_1(t)\right),$$

$$|\bar\partial\psi\cdot N|^2_3 \leq C_\delta\mathscr{M}_0 + \delta\mathscr{E}_0(t) + C_\delta\mathscr{P}\left(\sup_{t\in[0,T]}\mathscr{E}_0(t)\right).$$

Since $\mathscr{E}_k(t) \le \mathscr{E}(t)$, we obtain that

$$\sup_{t\in[0,T]} |\bar{\partial} v(t)\cdot N|^2_{1.5,\pm} \le C_\delta \mathscr{M}_0 + \delta \sup_{t\in[0,T]} \mathscr{E}(t) + C_\delta T \mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right), \tag{4.31}$$

and

$$\sup_{t\in[0,T]} |\bar{\partial} \psi(t)\cdot N|^2_{3} \le C_\delta \mathscr{M}_0 + \delta \sup_{t\in[0,T]} \mathscr{E}(t) + C_\delta T \mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right). \tag{4.32}$$

Estimates (4.30)–(4.32) then conclude Theorem 4.1. □

4.5. *A Uniform Bound for $\mathscr{E}(t)$*

Using (2.3), combining estimates (2.4) and (4.30)–(4.32), we find that for all $t \in [0, T]$,

$$\mathscr{E}(t) \le \mathscr{M}_0 + CT\mathscr{P}\left(\sup_{t\in[0,T]} \mathscr{E}(t)\right).$$

This is the polynomial inequality (2.5) that we had sought. It follows that by taking $T > 0$ sufficiently small,

$$\sup_{t\in[0,T]} \mathscr{E}(t) \le 2\mathscr{M}_0. \tag{4.33}$$

Finally, choose $T > 0$ even smaller so that the fundamental theorem of calculus ensures that the basic assumptions of Section 3.2 are satisfied.

5. The Limit as $\epsilon \to 0$

Having established our ϵ-independent estimate (4.33), we can now pass to the limit as $\epsilon \to 0$, and show that we recover the solutions of the one-phase Euler equations (1.4a).

Let $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ so that $\varphi \cdot N = 0$ on $\partial\mathscr{D}$. Testing (1.3) against φ, since φ is continuous across Γ, we find that

$$\begin{aligned} &\int_{\Omega^+} v_t^{+i} \varphi^i dx - \int_{\Omega^+} q^+ A_i^k \varphi^i_{,k} dx + \int_\Gamma (q^+ - q^-) A_i^k N_k \varphi^i dS \\ &\quad + \epsilon \int_{\Omega^-} v_t^{-i} \varphi^i dx + \epsilon \int_{\Omega^-} \mathrm{w}^j v^{-i}_{,j} \varphi^i dx - \int_{\Omega^-} q^- A_i^k \varphi^i_{,k} dx = 0. \end{aligned} \tag{5.1}$$

Note that v^+, v^-, q^+ and q^- above depend on ϵ implicitly. Our *a priori* bound (4.33) allows us to find sequences, still parameterized by ϵ, such that $\epsilon \to 0$,

$$v_t^+ \rightharpoonup \mathfrak{v}_t \quad \text{in } L^2(0, T; H^{1.5}(\Omega^+)), \tag{5.2a}$$

$$v^+ \to \mathfrak{v} \quad \text{in } L^2(0, T; H^2(\Omega^+)), \tag{5.2b}$$

$$v^+ \rightharpoonup \mathfrak{v} \quad \text{in } L^2(0, T; H^3(\Omega^+)), \tag{5.2c}$$

$$q^+ \rightharpoonup \mathfrak{q} \quad \text{in } L^2(0, T; H^{2.5}(\Omega^+)). \tag{5.2d}$$

Let $\tilde{\psi} = e + \int_0^t \mathfrak{v} ds$. By (5.2b), $\psi \to \tilde{\psi}$ in $L^\infty(0, T; H^2(\Omega^+))$; hence $A \to \mathscr{A} := (D\tilde{\psi})^{-1}$ in $L^\infty(0, T; H^1(\Omega^+))$ and $\Delta_g \psi \rightharpoonup \Delta_{\tilde{g}} \tilde{\psi}$ in $L^\infty(0, T; H^{0.5}(\Gamma))$ with $\tilde{g}_{\alpha\beta} = \tilde{\psi}_{,\alpha} \cdot \tilde{\psi}_{,\beta}$ because of (5.2c). Therefore, by (1.3d), (5.1) converges to

$$\int_{\Omega^+} \mathfrak{v}_t^i \varphi^i dx - \int_{\Omega^+} \mathfrak{q} \mathscr{A}_i^k \varphi^i_{,k} dx + \int_\Gamma \mathscr{H} \mathscr{A}_i^k N_k \varphi^i dS = 0,$$

where $\mathscr{H} = -\Delta_{\tilde{g}} \tilde{\psi} \cdot \tilde{n}$. This shows that $U = \mathfrak{v} \circ \tilde{\psi}^{-1}$ and $P = \mathfrak{q} \circ \tilde{\psi}^{-1}$ solve (1.4).

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References

[1] Ambrose, D.M. (2003). Well-posedness of vortex sheets with surface tension. *SIAM J. Math. Anal.* 35:211–244.

[2] Ambrose, D.M., Masmoudi, N. (2005). The zero surface tension limit of two-dimensional water waves. *Comm. Pure Appl. Math.* 58:1287–1315.

[3] Ambrose, D.M., Masmoudi, N. (2007). Well-posedness of 3D vortex sheets with surface tension. *Comm. Math. Sci.* 5:391–430.

[4] Cheng, C.H.A., Coutand, D., Shkoller, S. (2008). On the motion of vortex sheets with surface tension in the 3D Euler equations with vorticity. *Comm. Pure Appl. Math.* 61:1715–1752.

[5] Coutand, D., Shkoller, S. (2006). On the interaction between quasilinear elastodynamics and the Navier–Stokes equations. *Arch. Rational Mech. Anal.* 179:303–352.

[6] Coutand, D., Shkoller, S. (2007). Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.* 20:829–930.

[7] Craig, W. (1985). An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. Part. Diff. Eqs.* 10:787–1003.

[8] Dacorogna, B., Moser, J. (1990). On a partial differential equation involving the jacobian determinant. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 7:1–26.

[9] Kamotski, V., Lebeau, G. (2005). On 2D Rayleigh–Taylor instabilities. *Asymptot. Anal.* 42:1–27.

[10] Lannes, D. (2005). Well-posedness of the water-waves equations. *J. Amer. Math. Soc.* 18:605–654.

[11] Lindblad, H. (2005). Well-posedness for the motion of an incompressible liquid with free surface boundary. *Annals of Math.* 162:109–194.

[12] Nalimov, V.I. (1974). The Cauchy–Poisson problem. *Dynamika Splosh. Sredy* 18:104–210 (in Russian).

[13] Schweizer, B. (2005). On the three-dimensional Euler equations with a free boundary subject to surface tension. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 22:753–781.

[14] Shatah, J., Zeng, C. (2008). Geometry and a priori estimates for free boundary problems of the Euler's equation. *Comm. Pure Appl. Math.* 61:698–744.

[15] Shatah, J., Zeng, C. (2008). A priori estimates for fluid interface problems. *Comm. Pure Appl. Math.* 61:848–876.

[16] Taylor, M. (1991). *Partial Differential Equations*. Volume I: Basic Theory, New York: Springer Verlag.

[17] Temam, R. (2001). *Navier–Stokes Equations: Theory and Numerical Analysis*. Providence, RI: American Mathematical Society.

[18] Wu, S. (1997). Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.* 130:39–72.

[19] Wu, S. (1999). Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.* 12:445–495.
[20] Wu, S. (2006). Mathematical analysis of vortex sheets. *Comm. Pure Appl. Math.* 59:1065–1206.
[21] Yosihara, H. (1982). Gravity waves on the free surface of an incompressible perfect fluid. *Publ. RIMS Kyoto Univ.* 18:49–96.
[22] Yoshihra, H. (1983). Capillary-gravity waves for an incompressible ideal fluid. *J. Math. Kyoto Univ.* 23:649–694.