SOLVABILITY AND REGULARITY FOR AN ELLIPTIC SYSTEM PRESCRIBING THE CURL, DIVERGENCE, AND PARTIAL TRACE OF A VECTOR FIELD ON SOBOLEV-CLASS DOMAINS

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Contents

1. Introduction

1.1. **Th main results.** Given a Sobolev-class bounded domain $\Omega \subseteq \mathbb{R}^n$ and forcing functions $\boldsymbol{f}$ and g in Ω together with either h or $\boldsymbol{h}$ on $\partial\Omega$, we establish the solvability and regularity for solutions $\boldsymbol{v}$ to the following vector elliptic system of Hodge-type:

$$\begin{aligned} \operatorname{curl} \boldsymbol{v} &= \boldsymbol{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{div} \boldsymbol{v} &= g \quad \text{in} \quad \Omega\,, \end{aligned}$$

with boundary conditions given by either

$$\boldsymbol{v}\cdot \mathbf{N} = h \quad \text{or} \quad \boldsymbol{v}\times\mathbf{N} = \boldsymbol{h} \quad \text{on} \quad \partial\Omega\,.$$

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Motivated by the analysis of the free-boundary problems which arise in inviscid fluid dynamics, we provide a self-contained proof of the following two theorems:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded $H^{\mathrm{k}+1}$-domain with $\mathrm{k} > \frac{3}{2}$. Given $\boldsymbol{f}, g \in H^{\ell-1}(\Omega)$ with $\operatorname{div} \boldsymbol{f} = 0$, suppose that*

$$\operatorname{curl} \boldsymbol{v} = \boldsymbol{f} \quad \text{in} \quad \Omega\,, \tag{2a}$$

$$\operatorname{div} \boldsymbol{v} = g \quad \text{in} \quad \Omega\,. \tag{2b}$$

(1) If $h \in H^{\ell-0.5}(\partial\Omega)$ satisfies $\int_\Omega g\,dx = \int_{\partial\Omega} h\,dS$, and

$$\boldsymbol{v} \cdot \mathbf{N} = h \ \text{on} \ \partial\Omega\,, \tag{3}$$

then, for $1 \leqslant \ell \leqslant \mathrm{k}$, there exists a solution $\boldsymbol{v} \in H^\ell(\Omega)$ to (2) with boundary condition (3) such that

$$\|\boldsymbol{v}\|_{H^\ell(\Omega)} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|g\|_{H^{\ell-1}(\Omega)} + \|h\|_{H^{\ell-0.5}(\partial\Omega)}\Big].$$

(2) If $\boldsymbol{h} \in H^{\ell-0.5}(\partial\Omega)$ satisfies $\boldsymbol{h} \cdot \mathbf{N} = 0$ on $\partial\Omega$ and

$$\int_\Sigma \boldsymbol{f} \cdot \mathbf{N}\,dS = \oint_{\partial\Sigma} (\mathbf{N} \times \boldsymbol{h}) \cdot d\mathbf{r} \ \text{if} \ \Sigma \subseteq \partial\Omega \ \text{has piecewise smooth boundary},$$

and

$$\boldsymbol{v} \times \mathbf{N} = \boldsymbol{h} \ \text{on} \ \partial\Omega\,, \tag{4}$$

then, for $1 \leqslant \ell \leqslant \mathrm{k}$, there exists a solution $\boldsymbol{v} \in H^\ell(\Omega)$ to (2) with boundary condition (4) such that

$$\|\boldsymbol{v}\|_{H^\ell(\Omega)} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|g\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{h}\|_{H^{\ell-0.5}(\partial\Omega)}\Big].$$

The solution to either problem is unique if Ω is convex or if $\ell \geqslant 2$.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^{\mathrm{n}}$, $\mathrm{n} = 2$ or 3, be a bounded $H^{\mathrm{k}+1}$-domain with $\mathrm{k} > \frac{\mathrm{n}}{2}$. Then there exists a generic constant C depending on $|\partial\Omega|_{H^{\mathrm{k}+0.5}}$ such that for all $\boldsymbol{u} \in H^{\mathrm{k}+1}(\Omega)$,*

$$\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\operatorname{curl}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\nabla_{\partial\Omega}\boldsymbol{u} \cdot \mathbf{N}\|_{H^{\mathrm{k}-0.5}(\partial\Omega)}\Big], \tag{5}$$

$$\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\operatorname{curl}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\nabla_{\partial\Omega}\boldsymbol{u} \times \mathbf{N}\|_{H^{\mathrm{k}-0.5}(\partial\Omega)}\Big], \tag{6}$$

where $\nabla_{\partial\Omega}\boldsymbol{u}$ is the tangential derivative on $\partial\Omega$.

Remark 1.3. *The inequalities (5) and (6) play a fundamental role in the regularity theory of the Euler equations with moving interfaces; see, for example, [9] for the incompressible setting and [10] for the compressible problem with vacuum. The use of the norm $\|\nabla_{\partial\Omega}\boldsymbol{u} \cdot \mathbf{N}\|_{H^{\mathrm{k}-0.5}(\partial\Omega)}$ rather than $\|\boldsymbol{u} \cdot \mathbf{N}\|_{H^{\mathrm{k}+0.5}(\partial\Omega)}$ is crucial, as the regularity of the normal vector to field to $\partial\Omega$ is often worse than the regularity of the velocity vector $\boldsymbol{u}$.*

On the other hand, if Ω is at least of class H^{k+2} then the inequalities (5) and (6) can be replaced, respectively, by

$$\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\operatorname{curl}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\boldsymbol{u} \cdot \mathbf{N}\|_{H^{\mathrm{k}+0.5}(\partial\Omega)}\Big] \tag{7}$$

$$\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\operatorname{curl}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\boldsymbol{u} \times \mathbf{N}\|_{H^{\mathrm{k}+0.5}(\partial\Omega)}\Big] \tag{8}$$

Remark 1.4. *Recently, Amrouche & Seloula [5] established the inequality (7) in the L^p framework and for domains Ω of class $\mathscr{C}^{\mathrm{k}+1}$, under the additional assumption that $\boldsymbol{u} \times \mathbf{N} = 0$ on $\partial\Omega$. Similarly, they established (8) for $\mathscr{C}^{\mathrm{k}+1}$-class domains, under the additional assumption that $\boldsymbol{u} \cdot \mathbf{N} = 0$ on $\partial\Omega$.*

When Ω is very close to a $\mathscr{C}^\infty$-domain, we can obtain these inequalities for fractional-order Sobolev spaces, as in the following

Theorem 1.5. *Let $\Omega \subseteq \mathbb{R}^{\mathrm{n}}$, $\mathrm{n} = 2$ or 3, be a bounded H^{s+1}-domain with $s \in \mathbb{R}$ such that $s > \frac{\mathrm{n}}{2}$, and let $\mathcal{D}$ denote a $\mathscr{C}^\infty$-domain such that the distance between $\partial\mathcal{D}$ and $\partial\Omega$ in the $H^{s+0.5}$-norm is less than ϵ for $0 < \epsilon \ll 1$. Then there exists a generic constant C depending only on $|\partial\mathcal{D}|_{H^{s+0.5}}$, such that for all $\boldsymbol{u} \in H^{s+1}(\Omega)$,*

$$\|\boldsymbol{u}\|_{H^{s+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\mathrm{curl}\boldsymbol{u}\|_{H^s(\Omega)} + \|\mathrm{div}\boldsymbol{u}\|_{H^s(\Omega)} + \|\nabla_{\partial\Omega}\boldsymbol{u}\cdot\mathbf{N}\|_{H^{s-0.5}(\partial\Omega)}\Big], \tag{9}$$

$$\|\boldsymbol{u}\|_{H^{s+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\mathrm{curl}\boldsymbol{u}\|_{H^s(\Omega)} + \|\mathrm{div}\boldsymbol{u}\|_{H^s(\Omega)} + \|\nabla_{\partial\Omega}\boldsymbol{u}\times\mathbf{N}\|_{H^{s-0.5}(\partial\Omega)}\Big], \tag{10}$$

where $\nabla_{\partial\Omega}\boldsymbol{u}$ is the tangential derivative on $\partial\Omega$.

The inequalities (9) and (10) set in fractional-order Sobolev spaces are fundamental to the analysis of Euler-type free-boundary problems.

1.2. Outline of the paper.

In Section 2, we introduce our notation as well as a number of elementary technical lemmas, whose proofs we include (for completeness) in Appendix A. Section 3 is devoted to the analysis of the vector-valued elliptic system (28a) with mixed-type boundary conditions (28b) and (28c), which is fundamental to the proof of our two main theorems; in particular, we prove Theorem 3.5 which establishes the elliptic estimate for (28) when the coefficients are of Sobolev-class. As a corollary to this theorem, we state in Corollary 3.7 the basic elliptic estimates for both the Dirichlet and Neumann problems, again with Sobolev class regularity. Finally, for coefficients which are close to the identity, we give an improved estimate in Theorem 3.8 for solutions to (28), which is linear in the highest derivatives of the coefficient matrix. This latter theorem is essential for estimates in fractional-order Sobolev spaces via linear interpolation.

In Section 4, we prove Theorem 1.2, using the elliptic regularity theory developed for the elliptic system (28). Then, in Section 5, we prove Theorem 1.1. Our proof relies on some basic geometric identities involving the mean curvature of $\partial\Omega$, together with the elliptic regularity theory established in Section 3. Finally, in Section 6, we prove Theorem 1.5.

1.3. A brief history of prior results.

In addition to the recent work of Amrouche & Seloula [5] noted above, there have been many other methods and results to study such elliptic systems on smooth domains. The elliptic system (2) can be viewed as a particular example of the systems studied by Agmon, Douglis & Nirenberg [1], wherein both Schauder-type estimates and L^p-estimates can be found.

In [15], von Wahl proved that if the normal or the tangential trace of a vector field vanishes, and for bounded or unbounded Ω, the inequality $\|\nabla u\|_{L^p(\Omega)} \leqslant C\big(\|\mathrm{div}u\|_{L^p(\Omega)} + \|\mathrm{curl}u\|_{L^p(\Omega)}\big)$ is equivalent to the vanishing of the first Betti number.

Amrouche & Girault [4] derived the L^p-regularity theory of the steady Stokes equation by establishing the equivalency between the Sobolev space $W^{m,r}$ and the direct sum of $W^{m,r}$ by divergence-free vector fields and the gradients of $W^{m+1,r}$ functions.

Schwarz [14], studied the Hodge decomposition on manifolds with boundaries and showed that a differential k-forms can be written as the sum of an exact form, a coexact form, and a harmonic form.

Bolik & von Wahl [6] derived $\mathscr{C}^\alpha$-estimates of the gradient of a vector field whose curl, divergence, and normal or tangential traces are prescribed. Mitrea, Mitrea & Pipher [13] studied the vector potential theory on non-smooth domains in $\mathbf{R}^3$ with applications to electromagnetic scattering.

In [2], Amrouche, Bernardi, Dauge & Girault studied the vector potential associated with a divergence-free vector field satisfying various types of boundary conditions; see also Amrouche, Ciarlet & Ciarlet Jr. [3].

Buffa and Ciarlet Jr. [7] and [8] established the Hodge decomposition of tangential vector fields defined on polyhedron domains, and studied the tangential trace and tangential components of vectors belonging to the space $H(\text{curl},\Omega) := \{u \in L^2(\Omega;\mathbb{R}^3) \,|\, \text{curl} u \in L^2(\Omega;\mathbb{R}^3)\}$.

In [12], Kozono and Yanagisawa proved the decomposition of a divergence-free vector-field as the sum of the curl of a vector-field and a vector-field which is solenoidal, irrotational and has zero normal trace.

2. Notation and Preliminary Results

The Einstein summation convention is used throughout the paper. In particular, repeated Latin indices are summed from 1 to n, and repeated Greek indices are summed from 1 to n − 1. For example, $f_i g_i = \sum\limits_{i=1}^{\text{n}} f_i g_i$ and $f_\alpha g_\alpha = \sum\limits_{i=1}^{\text{n}-1} f_\alpha g_\alpha$.

2.1. H^s-domain.

In order to make our presentation self-contained, in this section, we collect a number of useful technical lemmas. These lemmas are well-known when the domains are smooth, but we shall need these basic results for Sobolev class domains. The proofs will be collected in Appendix A. For the remainder of this section, when not explicitly stated, s will denote a real number, while $0 \leqslant \text{k}, \ell$ will denote integers. We use the term domain to mean an open subset of $\mathbb{R}^{\text{n}}$.

Definition 2.1. *Let $\Omega \subseteq \mathbb{R}^{\text{n}}$ be a bounded domain, and $s > \frac{\text{n}}{2} + 1$ be a real number. Ω is said to be an H^s-domain, or of class H^s, if there exists a smooth bounded domain O and a map ψ such that $\psi : \text{O} \to \Omega$ is an H^s-diffeomorphism; that is,*

(1) *$\psi : \text{O} \to \Omega$ is one-to-one and onto, with differentiable inverse map $\psi^{-1} : \Omega \to \text{O}$, and*
(2) *$\psi \in H^s(\text{O})$ and $\psi^{-1} \in H^s(\Omega)$.*

By the trace theorem, $\psi|_{\partial\text{O}} \in H^{s-0.5}(\partial\text{O})$ and we shall often denote the value of this norm by $|\partial\Omega|_{H^{s-0.5}}$.

Definition 2.2. *For $s > \frac{\text{n}}{2} + 1$, a pair $(\mathcal{U}, \theta)$ is called a local chart of $\partial\Omega$ if $\mathcal{U} \subseteq \mathbb{R}^{\text{n}}$ is open, and $\theta : \mathbb{R}^{\text{n}-1} \cap B(0,1) \to \partial\Omega \cap \mathcal{U}$ is an H^s-diffeomorphism. The induced metric in the local chart $(\mathcal{U}, \theta)$ is the $(0,2)$-tensor $g_{\alpha\beta}$ given by*

$$g_{\alpha\beta} = \frac{\partial\theta}{\partial y_\alpha} \cdot \frac{\partial\theta}{\partial y_\beta}\,,$$

and the induced second-fundamental form in a local chart $(\mathcal{U}, \theta)$ is the $(0,2)$-tensor $b_{\alpha\beta}$ given by

$$b_{\alpha\beta} = -\frac{\partial^2\theta}{\partial y_\alpha \partial y_\beta} \cdot (\mathbf{N} \circ \theta^{-1})\,,$$

where $\mathbf{N}$ is the outward-pointing unit normal to $\partial\Omega$.

Definition 2.3. *For $s > \frac{\text{n}}{2} + 1$, let $\Omega \subseteq \mathbb{R}^{\text{n}}$ be a bounded H^s-domain. We let $\nabla_{\partial\Omega}$ denote the tangential derivative on $\partial\Omega$. If $u : \partial\Omega \to \mathbb{R}^n$ is differentiable, then in local chart $(\mathcal{U}, \theta)$, $\nabla_{\partial\Omega} u$ is given by*

$$(\nabla_{\partial\Omega} u) \circ \theta = \frac{\partial(u \circ \theta)}{\partial y_\alpha}\,,$$

2.2. **Basic inequalities.** We now state some basic inequalities, that we use throughout the paper.

Proposition 2.4. *For* $\mathrm{k} > \frac{\mathrm{n}}{2}$ *and* $0 \leqslant \ell \leqslant \mathrm{k}$*, let* $\mathrm{O} \subseteq \mathbb{R}^{\mathrm{n}}$ *be a bounded smooth domain. Then for all* $\epsilon \in \big(0, \frac{1}{4}\big)$*, there exists a constant* C_ϵ *depending on* ϵ *such that for all* $f \in H^{\mathrm{k}}(\mathrm{O})$ *and* $g \in H^{\ell}(\mathrm{O})$*,*

$$\sum_{j=1}^{\ell} \|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \leqslant C_\epsilon \|f\|_{H^{\mathrm{k}}(\mathrm{O})} \|g\|_{H^{\ell-\epsilon}(\mathrm{O})} \,. \tag{11}$$

Moreover, for some generic constant $C > 0$*,*

$$\|fg\|_{H^{\ell}(\mathrm{O})} \leqslant C \|f\|_{H^{\mathrm{k}}(\mathrm{O})} \|g\|_{H^{\ell}(\mathrm{O})} \qquad \forall\, f \in H^{\mathrm{k}}(\mathrm{O}), g \in H^{\ell}(\mathrm{O})\,. \tag{12}$$

Remark 2.5. *Suppose that* $s > \frac{\mathrm{n}}{2}$ *and* $0 \leqslant r \leqslant s$ *for some real numbers* r *and* s*. Then there exists a generic constant* $C_s > 0$ *such that*

$$\|fg\|_{H^r(\mathbb{R}^{\mathrm{n}})} \leqslant C_s \|f\|_{H^s(\mathbb{R}^{\mathrm{n}})} \|g\|_{H^r(\mathbb{R}^{\mathrm{n}})} \qquad \forall\, f \in H^s(\mathbb{R}^{\mathrm{n}}), g \in H^r(\mathbb{R}^{\mathrm{n}})\,. \tag{13}$$

By the Sobolev extension argument, we also conclude that

$$\|fg\|_{H^r(\Omega)} \leqslant C_s \|f\|_{H^s(\Omega)} \|g\|_{H^r(\Omega)} \qquad \forall\, f \in H^s(\Omega), g \in H^r(\Omega) \tag{14}$$

if Ω *is a bounded smooth domain.*

The following two corollaries are direct consequences of Proposition 2.4, and are the foundation of the study of inequalities on H^s-domains. The proof of these two corollaries can also be found in Appendix A.

Corollary 2.6. *Let* $\mathrm{O} \subseteq \mathbb{R}^{\mathrm{n}}$ *be a bounded smooth domain, and* $\psi : \mathrm{O} \to \Omega \subseteq \mathbb{R}^{\mathrm{n}}$ *be a* $H^{\mathrm{k}+1}$*-diffeomorphism for some integer* $\mathrm{k} > \frac{\mathrm{n}}{2}$*. Define* $J = \det(\nabla\psi)$ *and* $A = (\nabla\psi)^{-1}$*. Then*

$$\|J\|_{H^{\mathrm{k}}(\mathrm{O})} + \|A\|_{H^{\mathrm{k}}(\mathrm{O})} \leqslant C\big(\|\nabla\psi\|_{H^{\mathrm{k}}(\mathrm{O})}\big)\,. \tag{15}$$

Corollary 2.7. *Let* $\mathrm{O} \subseteq \mathbb{R}^{\mathrm{n}}$ *be a bounded smooth domain, and* $\psi : \mathrm{O} \to \Omega \subseteq \mathbb{R}^{\mathrm{n}}$ *be an* $H^{\mathrm{k}+1}$*-diffeomorphism for some integer* $\mathrm{k} > \frac{\mathrm{n}}{2}$*. Then for all* $0 \leqslant \ell \leqslant \mathrm{k}+1$*,*

$$\|f\|_{H^{\ell}(\Omega)} \leqslant C(\|\nabla\psi\|_{H^{\mathrm{k}}(\mathrm{O})}) \|f \circ \psi\|_{H^{\ell}(\mathrm{O})} \qquad \forall\, f \in H^{\ell}(\Omega)\,, \tag{16a}$$

$$\|f \circ \psi\|_{H^{\ell}(\mathrm{O})} \leqslant C(\|\nabla\psi\|_{H^{\mathrm{k}}(\mathrm{O})}) \|f\|_{H^{\ell}(\Omega)} \qquad \forall\, f \in H^{\ell}(\Omega)\,. \tag{16b}$$

By Corollary 2.7, if Ω is of class $H^{\mathrm{k}+1}$ with $\mathrm{k} > \frac{\mathrm{n}}{2}$, $f \in H^{\mathrm{k}}(\Omega)$ and $g \in H^{\ell}(\Omega)$, then $f \circ \psi \in H^{\mathrm{k}}(\mathrm{O})$ and $g \circ \psi \in H^{\ell}(\mathrm{O})$. As a consequence,

$$\begin{aligned} \|fg\|_{H^{\ell}(\Omega)} &\leqslant C \|(fg) \circ \psi\|_{H^{\ell}(\mathrm{O})} \\ &\leqslant C \|f \circ \psi\|_{H^{\mathrm{k}}(\mathrm{O})} \|g \circ \psi\|_{H^{\ell}(\mathrm{O})} \leqslant C \|f\|_{H^{\mathrm{k}}(\Omega)} \|g\|_{H^{\ell}(\Omega)} \end{aligned}$$

for some constant $C = C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})$. Similar arguments can be applied to show the following two propositions, and the proof is left to the reader.

Proposition 2.8. *Let* $\Omega \subseteq \mathbb{R}^{\mathrm{n}}$ *be a bounded domain of class* $H^{\mathrm{k}+1}$ *for some integer* $\mathrm{k} > \frac{\mathrm{n}}{2}$*. Then for all* $\epsilon \in \big(0, \frac{1}{4}\big)$*, there exists constant* C_ϵ *depending on* $|\partial\Omega|_{H^{\mathrm{k}+0.5}}$ *and* ϵ *such that for all* $f \in H^{\mathrm{k}}(\mathrm{O})$ *and* $g \in H^{\ell}(\mathrm{O})$*,* $0 \leqslant \ell \leqslant \mathrm{k}$*,*

$$\sum_{j=1}^{\ell} \|D^j f D^{\ell-j} g\|_{L^2(\Omega)} \leqslant C_\epsilon \|f\|_{H^{\mathrm{k}}(\Omega)} \|g\|_{H^{\ell-\epsilon}(\Omega)} \,. \tag{17}$$

Moreover, for some constant generic C depending on $|\partial\Omega|_{H^{k+0.5}}$,

$$\|fg\|_{H^\ell(\Omega)} \leqslant C\|f\|_{H^k(\Omega)}\|g\|_{H^\ell(\Omega)} \qquad \forall\, f \in H^k(\Omega), g \in H^\ell(\Omega)\,. \tag{18}$$

Proposition 2.9. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain of class H^{k+1} for some integer $k > \frac{n}{2}$, and $\psi : O \to \Omega \subseteq \mathbb{R}^n$ be an H^{k+1}-diffeomorphism. Then for all $0 \leqslant \ell \leqslant k$,*

$$\|f\|_{H^\ell(\Omega)} \leqslant C(|\partial\Omega|_{H^{k+0.5}})\|f\circ\psi\|_{H^\ell(O)} \qquad \forall\, f \in H^\ell(\Omega)\,, \tag{19a}$$

$$\|f\circ\psi\|_{H^\ell(O)} \leqslant C(|\partial\Omega|_{H^{k+0.5}})\|f\|_{H^\ell(\Omega)} \qquad \forall\, f \in H^\ell(\Omega)\,. \tag{19b}$$

Remark 2.10. *Note that Proposition 2.9 implies that the interpolation inequalities on a Sobolev class domain are still valid if the domain is bounded and has H^{k+1} regularity for some integer $k > \frac{n}{2}$. For example,*

$$\begin{aligned}\|f\|_{H^{0.5}(\Omega)} &\leqslant C(|\partial\Omega|_{H^{k+0.5}})\|f\circ\psi\|_{H^{0.5}(O)} \leqslant C(|\partial\Omega\|_{H^{k+0.5}})\|f\circ\psi\|^{\frac{1}{2}}_{L^2(O)}\|f\circ\psi\|^{\frac{1}{2}}_{H^1(O)} \\ &\leqslant C(|\partial\Omega|_{H^{k+0.5}})\|f\|^{\frac{1}{2}}_{L^2(\Omega)}\|f\|^{\frac{1}{2}}_{H^1(\Omega)}\,.\end{aligned}$$

The proofs of the following two lemmas are similar to the proof of Proposition 2.4, and are left to the reader.

Lemma 2.11. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded H^{k+1}-domain for some integer $k > \frac{n}{2}$.*

1. *Suppose that spt$(g) \subset\subset \Omega$. Then for $0 < \epsilon < \frac{1}{4}$ and $1 \leqslant \ell \leqslant k$,*

$$\big\|[\![\nabla^\ell, f]\!]g\big\|_{L^2(\Omega)} \leqslant C_\epsilon\|f\|_{H^k(\Omega)}\|g\|_{H^{\ell-\epsilon}(\Omega)}\,, \tag{20}$$

   where $[\![\nabla^\ell, f]\!]g = \nabla^\ell(fg) - f\nabla^\ell g$.
2. *Suppose that ζ is a smooth cut-off function such that*
   (a) *spt$(\zeta) \subseteq \mathcal{U}$;*
   (b) *there exists an H^{k+1}-diffeomorphism $\theta : B(0,1) \to \mathcal{U}$ satisfying*
      (i) *$\theta : B^+(0,1) \equiv B(0,1) \cap \{y_n > 0\} \to \mathcal{U} \cap \Omega$;*
      (ii) *$\theta : \{y_n = 0\} \to \partial\Omega$.*

   Define $F = (\zeta f)\circ\theta$ and $G = (\zeta g)\circ\theta$. Then for $0 < \epsilon < 1/4$, $1 \leqslant \ell \leqslant k$,

$$\big\|[\![\partial^\ell, F]\!]G\big\|_{L^2(B^+(0,r))} \leqslant C_\epsilon\|f\|_{H^k(\Omega)}\|g\|_{H^{\ell-\epsilon}(\Omega)}\,, \tag{21}$$

   where $[\![\partial^\ell, F]\!]G = \partial^\ell(FG) - F\partial^\ell G$ and $\partial = (\partial_{y_1}, \cdots, \partial_{y_{n-1}})$ denotes the tangential gradient.

Lemma 2.12. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded H^{k+1}-domain for some integer $k > \frac{n}{2}$. Then for each integers $\ell \in \{0,1\} \cup \big(\frac{n}{2}, k\big]$, there exists a generic constant $C = C(|\partial\Omega|_{H^{k+0.5}})$ such that*

$$\|fg\|_{H^\ell(\Omega)} \leqslant C\Big[\|f\|_{L^\infty(\Omega)}\|g\|_{H^\ell(\Omega)} + \|f\|_{H^\ell(\Omega)}\|g\|_{L^\infty(\Omega)}\Big] \quad \forall f, g \in H^\ell(\Omega) \cap L^\infty(\Omega)\,. \tag{22}$$

2.3. **Poincaré-type inequalities.** We will make use of the following Poincaré-type inequalities, whose proofs are similar to the proof of the standard Poincaré inequality, and are hence left to the reader.

Lemma 2.13. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded smooth domain with outward-pointing unit normal $\mathbf{N}$, and*

$$H^1_\tau(\Omega) \equiv \big\{u : \Omega \to \mathbb{R}^3 \,\big|\, u \in H^1(\Omega)\,, u \times \mathbf{N} = 0 \text{ on } \partial\Omega\big\}\,,$$
$$H^1_n(\Omega) \equiv \big\{u : \Omega \to \mathbb{R}^3 \,\big|\, u \in H^1(\Omega)\,, u \cdot \mathbf{N} = 0 \text{ on } \partial\Omega\big\}$$

are the collections of all vector-valued functions so that their tangential components or normal component vanishes on the boundary, respectively. Then

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leqslant C\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \qquad \forall\, \boldsymbol{u} \in H^1_\tau(\Omega)\,, \tag{23}$$

and

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leqslant C\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \qquad \forall\, \boldsymbol{u} \in H^1_n(\Omega)\,. \tag{24}$$

2.4. **Commutation with mollifiers.** Our proof of elliptic regularity relies on a mollification procedure (rather than the use of difference quotients).

Definition 2.14. *Let* $\eta(x) = C\exp\big(\frac{1}{|x|^2-1}\big)$ *for* $|x|<1$ *and* η *vanishes outside the unit ball, where* C *is chosen so that* $\|\eta\|_{L^1(\mathbb{R}^{\mathrm{n}})} = 1$*. The standard mollifier* η_ϵ *is defined by*

$$\eta_\epsilon(x) = \frac{1}{\epsilon^{\mathrm{n}}}\eta\big(\frac{x}{\epsilon}\big)\,.$$

We will make use of the following

Lemma 2.15. *For* $f \in W^{1,\infty}(\Omega)$ *and* $g \in L^2(\Omega)$ *with compact support, there is a generic constant* C *independent of* ϵ *such that*

$$\begin{aligned}\big\|D\big([\![\eta_\epsilon *, f]\!]g\big)\big\|_{L^2(\Omega)} &= \big\|D\big[\eta_\epsilon * (fg) - f\eta_\epsilon * g\big]\big\|_{L^2(\Omega)}\\ &\leqslant C\|f\|_{W^{1,\infty}(\Omega)}\|g\|_{L^2(\Omega)}\end{aligned} \tag{25}$$

for all $0<\epsilon<\min\big\{\mathrm{dist}\big(\partial\Omega,\mathrm{spt}(f)\big),\mathrm{dist}\big(\partial\Omega,\mathrm{spt}(g)\big)\big\}$*.*

Since we are dealing with problems on domains with boundaries, we make use of the *horizontal convolution-by-layers* operator, introduced in [9]. We define the horizontal convolution-by-layers operator Λ_ϵ as follows:

$$\Lambda_\epsilon f(x_h, x_{\mathrm{n}}) = \int_{\mathbb{R}^{\mathrm{n}-1}} \rho_\epsilon(x_h - y_h) f(y_h, x_{\mathrm{n}})dy_h \;\; \text{for } f(\cdot, x_{\mathrm{n}}) \in L^1(\mathbb{R}^{\mathrm{n}-1})\,,$$

where $\rho_\epsilon(x_h) = \frac{1}{\epsilon^{\mathrm{n}-1}}\rho\big(\frac{x_h}{\epsilon}\big)$, and $\rho \in C^\infty_0(\mathbb{R}^2)$ is given by $\rho(x) = C\exp\Big(\frac{1}{|x|^2-1}\Big)$ if $|x|<1$ and $\rho(x)=0$ if $|x_h| \geqslant 1$. The constant C is chosen so that $\int_{\mathbb{R}^{\mathrm{n}-1}} \rho dx = 1$. It follows that for $\epsilon > 0$, $0 \leqslant \rho_\epsilon \in C^\infty_0(\mathbb{R}^{\mathrm{n}-1})$ with $\mathrm{spt}(\rho_\epsilon) \subset \overline{B(0,\epsilon)}$. (Here, spt stands for support.)

It should be clear that Λ_ϵ smooths functions defined on $\mathbb{R}^{\mathrm{n}}$ along all horizontal subspaces, but does not smooth functions in the vertical x_{n}-direction. On the other hand, we can restrict the operator Λ_ϵ to act on functions $f : \mathbb{R}^{\mathrm{n}-1} \to \mathbb{R}$ as well, in which case Λ_ϵ becomes the usual mollification operator. Associated to Λ_ϵ, we need the following

Lemma 2.16. *For* $f \in W^{1,\infty}(\mathbb{R}^{\mathrm{n}}_+)$ *and* $g \in L^2(\mathbb{R}^{\mathrm{n}}_+)$*, there is a generic constant* C *independent of* ϵ *such that*

$$\big\|\partial\big([\![\Lambda_\epsilon, f]\!]g\big)\big\|_{L^2(\mathbb{R}^{\mathrm{n}}_+)} = \big\|\partial\big[\Lambda_\epsilon(fg) - f\Lambda_\epsilon g\big]\big\|_{L^2(\mathbb{R}^{\mathrm{n}}_+)} \leqslant C\|f\|_{W^{1,\infty}(\mathbb{R}^{\mathrm{n}}_+)}\|g\|_{L^2(\mathbb{R}^{\mathrm{n}}_+)} \tag{26}$$

for all $\epsilon > 0$*.*

2.5. **The Piola Identity.**

Lemma 2.17 (Piola identity)**.** *Let* $\psi : \Omega \subseteq \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}}$ *be a diffeomorphism, and* $[a_{ij}]_{\mathrm{n}\times\mathrm{n}}$ *be the cofactor matrix of* $\nabla\psi$*. Then*

$$\frac{\partial}{\partial x_j}\, a_{ji} = 0. \tag{27}$$

The proof can be found in [11].

3. Vector-Valued Elliptic Equations

Let $\Omega \subseteq \mathbb{R}^{\rm n}$ denote a bounded domain. In this section, we study a vector-valued elliptic equation

$$(L\boldsymbol{u})^i = \boldsymbol{u}^i - \frac{\partial}{\partial x_j}\Big(a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\Big) = \boldsymbol{f}^i \qquad \text{in} \quad \Omega\,, \tag{28a}$$

with special types of boundary conditions, where $\boldsymbol{u} = (\boldsymbol{u}_1, \cdots, u_{\rm n})$ and $f = (f_1, \cdots, f_{\rm n})$ are vector-valued functions, and a^{jk} is a two-tensor satisfying the positivity condition

$$a^{jk}\xi_j\xi_k \geqslant \lambda|\xi|^2 \qquad \forall\, \xi, \eta \in \mathbb{R}^{\rm n} \tag{29}$$

for some $\lambda > 0$. Since $\boldsymbol{u} \in \mathbb{R}^{\rm n}$, n boundary conditions are needed to solve the system uniquely.

We consider a mixed-type boundary condition given by

$$\boldsymbol{u}\cdot\mathbf{w} = 0 \qquad \text{on} \quad \partial\Omega\,, \tag{28b}$$

$$\mathrm{P}_{\mathbf{w}^\perp}\Big(a^{jk}\frac{\partial \boldsymbol{u}}{\partial x_k}\mathbf{N}_j - \boldsymbol{g}\Big) = \mathbf{0} \qquad \text{on} \quad \partial\Omega\,, \tag{28c}$$

where $\mathbf{w}$ is a uniformly continuous vector field defined in a neighborhood of $\partial\Omega$ which vanishes nowhere on $\partial\Omega$, $\mathrm{P}_{\mathbf{w}^\perp} : \mathbb{R}^{\rm n} \to \mathbb{R}^{\rm n}$ is the projection map given by

$$\mathrm{P}_{\mathbf{w}^\perp}(\boldsymbol{v}) = \boldsymbol{v} - \frac{(\boldsymbol{v}\cdot\mathbf{w})}{|\mathbf{w}|^2}\,\mathbf{w} = \big(\mathrm{Id} - \frac{\mathbf{w}\otimes\mathbf{w}}{|\mathbf{w}|^2}\big)\boldsymbol{v}\,. \tag{30}$$

The condition (28b) specifies the component of the vector $\boldsymbol{u}$ in the direction of $\mathbf{w}$, while (with $\mathbf{N}$ denoting the outward-pointing unit norm of $\partial\Omega$) the condition (28c) specifies the $\mathrm{n}-1$ components of the Neumann derivative $a^{jk}\dfrac{\partial \boldsymbol{u}^i}{\partial x_k}\mathbf{N}_j$.

Integrating by parts in x_j, we find that

$$\begin{aligned}
-\int_\Omega \frac{\partial}{\partial x_j}\Big(a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\Big)\boldsymbol{\varphi}^i dx &= \int_\Omega a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}dx - \int_{\partial\Omega} a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\mathbf{N}_j\boldsymbol{\varphi}^i dx \\
&= \int_\Omega a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}dx - \int_{\partial\Omega}\Big[\boldsymbol{g}^i + a^{jk}\frac{\partial \boldsymbol{u}^r}{\partial x_k}\mathbf{N}_j\frac{\mathbf{w}_r\mathbf{w}_i}{|\mathbf{w}|^2}\Big]\boldsymbol{\varphi}^i dx + \int_{\partial\Omega}\frac{\boldsymbol{g}\cdot\mathbf{w}}{|\mathbf{w}|^2}(\varphi\cdot\mathbf{w})\,dx\,.
\end{aligned}$$

The identity above motivates the following

Definition 3.1. *Let* $\mathcal{V} = \big\{\boldsymbol{v} \in H^1(\Omega)\,\big|\,\boldsymbol{v}\cdot\mathbf{w} = 0$ *on* $\partial\Omega\big\}$. *A function* $\boldsymbol{u} \in \mathcal{V}$ *is called a weak solution to* (28) *if*

$$(\boldsymbol{u}, \boldsymbol{\varphi})_{L^2(\Omega)} + \int_\Omega a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\,dx = (\boldsymbol{f}, \boldsymbol{\varphi})_{L^2(\Omega)} + \langle \boldsymbol{g}, \boldsymbol{\varphi}\rangle_{\partial\Omega} \quad \forall\, \boldsymbol{\varphi} \in \mathcal{V}\,, \tag{31}$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega}$ *denotes the duality pairing for functions defined on* $\partial\Omega$.

With the help of the Lax-Milgram theorem it is easy to conclude the following

Theorem 3.2. *Suppose that* $a^{jk} \in L^\infty(\Omega)$ *satisfies the ellipticity condition* (29), *and* $\mathbf{w}$ *is a uniformly continuous vector field defined in a neighborhood of* $\partial\Omega$ *which vanishes nowhere on* $\partial\Omega$. *Then for all* $\boldsymbol{f} \in L^2(\Omega)$ *and* $\boldsymbol{g} \in H^{-0.5}(\partial\Omega)$, *there exists a unique weak solution to* (28) *in* $\mathcal{V}$, *and the weak solution* $\boldsymbol{u}$ *satisfies*

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leqslant C\Big[\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{H^{-0.5}(\partial\Omega)}\Big]\,. \tag{32}$$

Remark 3.3. *Let $\boldsymbol{u} \in H^2(\Omega) \cap \mathcal{V}$ be a weak solution to (28). Integrating by parts in x_j, we find that*

$$\int_\Omega \Big(\boldsymbol{u}^i - \frac{\partial}{\partial x_j}\big(a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\big) - \boldsymbol{f}^i\Big)\boldsymbol{\varphi}^i dx + \int_{\partial\Omega}\big(a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\mathbf{N}_j - \boldsymbol{g}\big)\boldsymbol{\varphi}^i dS = 0 \quad \forall\, \boldsymbol{\varphi} \in \mathcal{V}\,.$$

Since $\boldsymbol{\varphi}\cdot\mathbf{w} = 0$ on $\partial\Omega$, we can only conclude (28c).

We next establish the regularity theory for functions satisfying (31).

3.1. The case that the coefficients a^{jk} are of class $\mathscr{C}^k$.

Theorem 3.4. *Suppose that for some $\mathrm{k} \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^{\mathrm{n}}$ is a bounded $\mathscr{C}^{\mathrm{k}+1}$-domain, $a^{jk} \in \mathscr{C}^{\mathrm{k}}(\overline{\Omega})$ satisfies the ellipticity condition (29), and $\mathbf{w}$ is $\mathscr{C}^{\mathrm{k}+1}$ in a neighborhood of $\partial\Omega$ which vanishes nowhere on $\partial\Omega$. Then for all $\boldsymbol{f} \in H^{\mathrm{k}-1}(\Omega)$ and $\boldsymbol{g} \in H^{\mathrm{k}-0.5}(\partial\Omega)$, the weak solution $\boldsymbol{u}$ to (28) in fact belongs to $H^{\mathrm{k}+1}(\Omega)$, and satisfies*

$$\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{f}\|_{H^{\mathrm{k}-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\mathrm{k}-0.5}(\partial\Omega)}\Big] \tag{33}$$

for some constant C depending on $\|a\|_{\mathscr{C}^{\mathrm{k}}(\Omega)}$, $\|\mathbf{w}\|_{\mathscr{C}^{\mathrm{k}+1}(\Omega)}$ and $|\partial\Omega|_{\mathscr{C}^{\mathrm{k}+1}}$.

Proof. The goal is to show that the function $\boldsymbol{u} \in \mathcal{V}$ satisfying the weak formulation (31) satisfies (33). We prove by induction and divide the proof into several steps as follows:
Step 1: Suppose that $\boldsymbol{u} \in H^\ell(\Omega)$ for some $1 \leqslant \ell \leqslant \mathrm{k}-1$. Let χ be a smooth function with $\mathrm{spt}(\chi) \subset\subset \Omega$, and $0 < \epsilon < \mathrm{dist}(\mathrm{spt}(\chi), \partial\Omega)$. Define

$$\boldsymbol{\varphi} = (-1)^\ell \chi\big[\eta_\epsilon * D^{2\ell}\big(\eta_\epsilon * (\chi\boldsymbol{u})\big)\big]$$

in which repeated ℓ does not mean summation over ℓ (since the range of ℓ is not 1 to n) but purely an index. Then $\boldsymbol{\varphi} \in \mathcal{V}$; thus $\boldsymbol{\varphi}$ can be used as a test function in (31). First we note that

$$(\boldsymbol{u}, \boldsymbol{\varphi})_{L^2(\Omega)} = \big\|D^\ell \eta_\epsilon * (\chi\boldsymbol{u})\big\|^2_{L^2(\Omega)}\,. \tag{34}$$

By the properties of convolution and the Leibniz rule, we find that

$$\begin{aligned}\int_\Omega a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}dx = &\int_\Omega D\big[\eta_\epsilon * \big(a^{jk}D^{\ell-1}(\chi\boldsymbol{u}^i),_k\big)\big]D^\ell\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_j\big]dx \\ &+ \sum_{r=0}^{\ell-2}\binom{\ell-1}{r}\int_\Omega D\big[\eta_\epsilon * \big((D^{\ell-1-r}a^{jk})D^r(\chi\boldsymbol{u}^i),_k\big)\big]D^\ell\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_j\big]dx \\ &- \int_\Omega D^\ell\big[\eta_\epsilon * \big(a^{jk}\boldsymbol{u}^i\chi,_k\big)\big]D^\ell\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_j\big]dx \\ &- \int_\Omega D^{\ell-1}\big[\eta_\epsilon * \big(a^{jk}\chi,_j\,\boldsymbol{u}^i,_k\big)\big]D^{\ell+1}\big[\eta_\epsilon * (\chi\boldsymbol{u}^i)\big]dx\,.\end{aligned} \tag{35}$$

Using the commutator notation, the first term on the right-hand side of the identity above can be rewritten as

$$\begin{aligned}&\int_\Omega D\big[\eta_\epsilon * \big(a^{jk}D^{\ell-1}(\chi\boldsymbol{u}^i),_k\big)\big]D^\ell\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_j\big]dx = \int_\Omega D\big[a^{jk}\eta_\epsilon * D^{\ell-1}(\chi\boldsymbol{u}^i),_k\big]D^\ell\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_j\big]dx \\ &\qquad + \int_\Omega \big[D[\![\eta_\epsilon *, a^{jk}]\!]D^{\ell-1}(\chi\boldsymbol{u}^i),_k\big]D^\ell\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_j\big]dx \\ &\quad = \int_\Omega a^{jk}D^\ell\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_k\big]D^\ell\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_j\big]dx + \int_\Omega (Da^{jk})D^{\ell-1}\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_k\big]D^\ell\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_j\big]dx \\ &\qquad + \int_\Omega \big[D[\![\eta_\epsilon *, a^{jk}]\!]D^{\ell-1}(\chi\boldsymbol{u}^i),_k\big]D^\ell\big[\eta_\epsilon * (\chi\boldsymbol{u}^i),_j\big]dx\,;\end{aligned}$$

thus after rearranging terms, the ellipticity condition implies that

$$\lambda\big\|D^{\ell+1}\big(\eta_\epsilon\ast(\chi\boldsymbol{u})\big)\big\|^2_{L^2(\Omega)} \leqslant \int_\Omega a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\,dx - \int_\Omega (Da^{jk})D^{\ell-1}\big[\eta_\epsilon\ast(\chi\boldsymbol{u}^i),_k\big]D^\ell\big[\eta_\epsilon\ast(\chi\boldsymbol{u}^i),_j\big]dx$$
$$- \int_\Omega \big[D\big[\!\big[\eta_\epsilon\ast, a^{jk}\big]\!\big]D^{\ell-1}(\chi\boldsymbol{u}^i),_k\big]D^\ell\big[\eta_\epsilon\ast(\chi\boldsymbol{u}^i),_j\big]dx$$
$$- \sum_{r=0}^{\ell-2}\binom{\ell-1}{r}\int_\Omega D\big[\eta_\epsilon\ast\big((D^{\ell-1-r}a^{jk})D^r(\chi\boldsymbol{u}^i),_k\big)\big]D^\ell\big[\eta_\epsilon\ast(\chi\boldsymbol{u}^i),_j\big]dx$$
$$+ \int_\Omega D^\ell\big[\eta_\epsilon\ast\big(a^{jk}\boldsymbol{u}^i\chi,_k\big)\big]D^\ell\big[\eta_\epsilon\ast(\chi\boldsymbol{u}^i),_j\big]dx + \int_\Omega D^{\ell-1}\big[\eta_\epsilon\ast\big(a^{jk}\chi,_j\,\boldsymbol{u}^i,_k\big)\big]D^{\ell+1}\big[\eta_\epsilon\ast(\chi\boldsymbol{u}^i)\big]dx\,.$$

The last five integrals on the right-hand side of the inequality above can be estimated using Hölder's inequality and the commutation estimate (25), and we obtain that

$$\lambda\big\|D^{\ell+1}\big(\eta_\epsilon\ast(\chi\boldsymbol{u})\big)\big\|^2_{L^2(\Omega)} \leqslant \int_\Omega a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\,dx + C\|a\|_{\mathscr{C}^\ell(\overline{\Omega})}\|\boldsymbol{u}\|_{H^\ell(\Omega)}\big\|D^{\ell+1}\big(\eta_\epsilon\ast(\chi\boldsymbol{u}^i)\big)\big\|_{L^2(\Omega)}\,. \tag{36}$$

On the other hand, it is easy to see that

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx = -\int_\Omega D^{\ell-1}\big[\eta_\epsilon\ast(\chi\boldsymbol{f}^i)\big]D^{\ell+1}\big[\eta_\epsilon\ast(\chi\boldsymbol{u}^i)\big]dx \leqslant C\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)}\big\|D^{\ell+1}\big(\eta_\epsilon\ast(\chi\boldsymbol{u})\big)\big\|_{L^2(\Omega)}\,. \tag{37}$$

Summing (34), (36) and (37), we find that

$$\big\|D^\ell(\eta_\epsilon\ast(\chi\boldsymbol{u})\big\|^2_{L^2(\Omega)} + \lambda\big\|D^{\ell+1}\big(\eta_\epsilon\ast(\chi\boldsymbol{u})\big)\big\|^2_{L^2(\Omega)}$$
$$\leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|a\|_{\mathscr{C}^\ell(\overline{\Omega})}\|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big]\big\|D^{\ell+1}\big(\eta_\epsilon\ast(\chi\boldsymbol{u})\big)\big\|_{L^2(\Omega)}\,;$$

thus by Young's inequality,

$$\big\|D^\ell(\eta_\epsilon\ast(\chi\boldsymbol{u})\big\|^2_{L^2(\Omega)} + \lambda\big\|D^{\ell+1}\big(\eta_\epsilon\ast(\chi\boldsymbol{u})\big)\big\|^2_{L^2(\Omega)}$$
$$\leqslant \frac{C}{\lambda}\Big[\|\boldsymbol{f}\|^2_{H^{\ell-1}(\Omega)} + \|a\|^2_{\mathscr{C}^\ell(\overline{\Omega})}\|\boldsymbol{u}\|^2_{H^\ell(\Omega)}\Big] + \frac{\lambda}{2}\big\|D^{\ell+1}\big[\eta_\epsilon\ast(\chi\boldsymbol{u}^i)\big]\big\|_{L^2(\Omega)}$$

which further implies that

$$\big\|D^{\ell+1}\big(\eta_\epsilon\ast(\chi\boldsymbol{u})\big)\big\|_{L^2(\Omega)} \leqslant \frac{C}{\lambda}\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|a\|_{\mathscr{C}^\ell(\overline{\Omega})}\|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big]\,. \tag{38}$$

Since $f \in H^{\mathrm{k}-1}(\Omega)$ and $a \in \mathscr{C}^{\mathrm{k}}(\overline{\Omega})$, the assumption that $\boldsymbol{u} \in H^\ell(\Omega)$ implies that the right-hand side of (38) is bounded independent of the smoothing parameter ϵ. Therefore, we can pass $\epsilon \to 0$ in (38) and obtain that

$$\|D^{\ell+1}(\chi\boldsymbol{u})\|_{L^2(\Omega)} \leqslant \frac{C}{\lambda}\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|a\|_{\mathscr{C}^\ell(\overline{\Omega})}\|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big]$$

or

$$\|\chi D^{\ell+1}u\|_{L^2(\Omega)} \leqslant \frac{C}{\lambda}\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \big(\|a\|_{\mathscr{C}^\ell(\overline{\Omega})} + \lambda\big)\|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big]\,. \tag{39}$$

This implies that $\boldsymbol{u} \in H^{\ell+1}_{\mathrm{loc}}(\Omega)$. In particular, since $\boldsymbol{u} \in H^1(\Omega)$ by the nature of being a weak solution, we must have $\boldsymbol{u} \in H^2_{\mathrm{loc}}(\Omega)$; thus we can integrate the weak formulation (31) by parts and find that

$$\int_\Omega \Big[\boldsymbol{u} - \frac{\partial}{\partial x_j}\Big(a^{jk}\frac{\partial \boldsymbol{u}}{\partial x_k}\Big) - \boldsymbol{f}\Big]\cdot\boldsymbol{\varphi}\,dx = 0 \qquad \forall\,\boldsymbol{\varphi} \in \mathscr{C}^\infty_0(\Omega)\,.$$

The above identity implies that

$$\boldsymbol{u}^i - \frac{\partial}{\partial x_j}\Big(a^{jk}\frac{\partial \boldsymbol{u}^i}{\partial x_k}\Big) = \boldsymbol{f}^i \quad \text{a.e. in } \Omega. \tag{40}$$

Step 2: Assume that $\boldsymbol{u} \in H^\ell(\Omega)$ for some $1 \leqslant \ell \leqslant \mathrm{k}-1$. Let $\{\mathcal{U}_m\}_{m=1}^K$ denote an open cover of Ω which intersects the boundary $\partial\Omega$, and let $\{\theta_m\}_{m=1}^K$ denote a collection of charts such that

(1) $\theta_m : B(0, r_m) \to \mathcal{U}_m$ is a $\mathscr{C}^\infty$-diffeomorphism,
(2) $\det(D\theta_m) = 1$,
(3) $\theta_m : B(0, r_m) \cap \{x_n = 0\} \to \mathcal{U}_m \cap \partial\Omega$,
(4) $\theta_m : B_m^+ \equiv B(0, r_m) \cap \{y_{\rm n} > 0\} \to \mathcal{U}_m \cap \Omega$.
(5) $\|\nabla\theta_m - \mathrm{Id}\|_{L^\infty(B(0,r_m))} \ll 1$.

Let $0 \leqslant \zeta_m \leqslant 1$ in $\mathscr{C}_0^\infty(\mathcal{U}_m)$ denote a partition of unity subordinate to the open covering $\mathcal{U}_m$; that is,

$$\sum_{m=0}^K \zeta_m = 1 \quad \text{and} \quad \mathrm{spt}(\zeta_m) \subseteq \mathcal{U}_m \quad \forall\, m\,.$$

Define a vector-valued function $\boldsymbol{\varphi}$ by

$$\boldsymbol{\varphi}^i = (-1)^\ell \big[\widetilde{\zeta}_m \widetilde{\mathbf{w}}_m^i \Lambda_\epsilon \partial^{2\ell} \Lambda_\epsilon(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)\big] \circ \theta_m^{-1} = (-1)^\ell \big[\widetilde{\zeta}_m \widetilde{\mathbf{w}}_m^i \Lambda_\epsilon \partial^{2\ell} \Lambda_\epsilon(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m^j \widetilde{\mathbf{w}}_m^j)\big] \circ \theta_m^{-1}\,,$$

where $\widetilde{\zeta}_m = \zeta_m \circ \theta_m$, $\widetilde{\mathbf{w}}_m = \mathbf{w} \circ \theta_m$, Λ_ϵ is the horizontal convolution operator, $\widetilde{\boldsymbol{u}}_m = \boldsymbol{u} \circ \theta_m$, and ∂ denotes the tangential gradient; that is, $\partial = (\partial_{y_1}, \cdots, \partial_{y_{\rm n-1}})$. Since $\boldsymbol{\varphi} \cdot \mathbf{w} = 0$ on $\partial\Omega$, $\boldsymbol{\varphi} \in \mathcal{V}$ and can be used as a test function. The use of $\boldsymbol{\varphi}$ as a test function in (31) implies that

$$(\boldsymbol{u}, \boldsymbol{\varphi})_{L^2(\Omega)} + \int_\Omega a^{jk} \frac{\partial \boldsymbol{u}^i}{\partial x_k} \frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\, dx \leqslant C\Big[\|\boldsymbol{f}\|_{H^\ell(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)}\Big] \big\|\partial^\ell \Lambda_\epsilon(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)\big\|_{H^1(B_m^+)}\,,$$

where $B_m^+ = B(0, r_m) \cap \{y_{\rm n} > 0\}$. Similar to (34), it should be clear that

$$(\boldsymbol{u}, \boldsymbol{\varphi})_{L^2(\Omega)} = \big\|\partial^\ell \Lambda_\epsilon(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)\big\|^2_{L^2(B_m^+)}\,,$$

so now we proceed to the second term on the left-hand side.

Let $A = (\nabla\theta_m)^{-1}$ and $b^{rs} = (a^{jk} \circ \theta_m) A_k^s A_j^r$. We claim that b^{rs} satisfies the ellipticity condition. In fact, since $\|\nabla\theta_m - \mathrm{Id}\|_{L^\infty(B_m^+)} \ll 1$, we find that

$$b^{rs}\xi_r\xi_s = (a^{jk} \circ \theta_m) A_k^s A_j^r \xi_r \xi_s \geqslant \lambda |A^{\rm T}\xi|^2 \geqslant \frac{\lambda}{2}|\xi|^2\,. \tag{41}$$

Making change of variable formula $x = \theta_m(y)$ and then integrating by parts, we obtain that

$$\begin{aligned}
&\int_\Omega a^{jk} \frac{\partial \boldsymbol{u}^i}{\partial x_k} \frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\, dx = \int_{B_m^+} b^{rs} \frac{\partial \widetilde{\boldsymbol{u}}_m^i}{\partial y_r} \frac{\partial \boldsymbol{\varphi}^i}{\partial y_s}\, dy \\
&\quad = \int_{B_m^+} \partial^\ell \Lambda_\epsilon\big(b^{rs} \widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_{m,s} \cdot \widetilde{\mathbf{w}}_m\big) \partial^\ell \Lambda_\epsilon(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)_{,r}\, dy \\
&\quad = \int_{B_m^+} \partial \Lambda_\epsilon\big(b^{rs} \partial^{\ell-1}(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)_{,s}\big) \partial^\ell \Lambda_\epsilon(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)_{,r}\, dy \\
&\qquad + \sum_{k=0}^{\ell-2} \binom{\ell-1}{k} \int_{B_m^+} \partial \Lambda_\epsilon\big(\partial^{\ell-1-k} b^{rs} \partial^k (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)_{,s}\big) \partial^\ell \Lambda_\epsilon(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)_{,r}\, dy \\
&\qquad - \int_{B_m^+} \partial^\ell \Lambda_\epsilon\big(b^{rs} \widetilde{\boldsymbol{u}}_m^i (\widetilde{\zeta}_m \widetilde{\mathbf{w}}_m^i)_{,s}\big) \partial^\ell \Lambda_\epsilon(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)_{,r}\, dy\,.
\end{aligned} \tag{42}$$

For the first term on the right-hand side of (42), as in Step 1 we use the commutator notation, and find that

$$\begin{aligned}
&\int_{B_m^+} \partial\Lambda_\epsilon\big(b^{rs}\partial^{\ell-1}(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_s\big)\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_r\,dy\\
&\quad=\int_{B_m^+}\partial\big[b^{rs}\Lambda_\epsilon\partial^{\ell-1}(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_s\big]\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_r\,dy\\
&\qquad+\int_{B_m^+}\big(\partial[\![\Lambda_\epsilon,b^{rs}]\!]\partial^{\ell-1}(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_s\big)\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_r\,dy\\
&\quad=\int_{B_m^+} b^{rs}\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_s\,\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_r\,dy\\
&\qquad+\int_{B_m^+}(\partial b^{rs})\Lambda_\epsilon\partial^{\ell-1}(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_s\,\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_r\,dy\\
&\qquad+\int_{B_m^+}\big(\partial[\![\Lambda_\epsilon,b^{rs}]\!]\partial^{\ell-1}(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_s\big)\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_r\,dy\,;
\end{aligned}$$

thus (41) and the commutation estimate (26) suggest that

$$\begin{aligned}
&\int_{B_m^+}\partial\Lambda_\epsilon\big(b^{rs}\partial^{\ell-1}(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_s\big)\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_r\,dy\\
&\qquad\geqslant\frac{\lambda}{2}\big\|\partial^\ell\Lambda_\epsilon\nabla(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m)\big\|^2_{L^2(B_m^+)}-C\|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big[\big\|\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m)\big\|_{H^1(B_m^+)}+\|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big].
\end{aligned}$$

For the remaining terms on the right-hand side of (42), we apply Hölder's inequality and find that

$$\begin{aligned}
&\sum_{k=0}^{\ell-2}\binom{\ell-1}{k}\int_{B_m^+}\partial\Lambda_\epsilon\big(\partial^{\ell-1-k}b^{rs}\partial^k(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_s\big)\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_r\,dy\\
&\qquad-\int_{B_m^+}\partial^\ell\Lambda_\epsilon\big(b^{rs}\widetilde{\boldsymbol{u}}^i_m(\widetilde{\zeta}_m\widetilde{\mathbf{w}}^i_m),_s\big)\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m),_r\,dy\\
&\qquad\qquad\leqslant C\|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big[\big\|\partial^\ell\Lambda_\epsilon\nabla(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m)\big\|_{L^2(B_m^+)}+\|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big],
\end{aligned}\tag{43}$$

where C depends on $\|a\|_{\mathscr{C}^\ell(\Omega)}$, $\|\mathbf{w}\|_{\mathscr{C}^{\ell+1}(\Omega)}$ and $|\partial\Omega|_{\mathscr{C}^{\ell+1}(\Omega)}$. As a consequence, by Young's inequality we conclude that

$$\begin{aligned}
&\big\|\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m)\big\|^2_{L^2(B_m^+)}+\lambda\big\|\partial^\ell\Lambda_\epsilon\nabla(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m)\big\|^2_{L^2(B_m^+)}\\
&\qquad\leqslant C_\delta\Big[\|\boldsymbol{u}\|^2_{H^\ell(\Omega)}+\|\boldsymbol{f}\|^2_{H^{\ell-1}(\Omega)}+\|\boldsymbol{g}\|^2_{H^{\ell-1.5}(\partial\Omega)}\Big]+\delta\big\|\partial^\ell\Lambda_\epsilon\nabla(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m)\big\|^2_{L^2(B_m^+)}
\end{aligned}$$

which, by choosing $\delta>0$ small enough, further suggests that

$$\big\|\partial^\ell\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m)\big\|_{H^1(B_m^+)}\leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)}+\|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)}+\|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big]$$

for some constant $C=C(\|a\|_{\mathscr{C}^\ell(\Omega)},\|\mathbf{w}\|_{\mathscr{C}^{\ell+1}(\Omega)},|\partial\Omega|_{\mathscr{C}^{\ell+1}})$.

Since the estimate above is independent of the smoothing parameter ϵ, by passing $\epsilon\to0$ we conclude that

$$\big\|\widetilde{\zeta}_m\partial^\ell(\widetilde{\boldsymbol{u}}_m\cdot\widetilde{\mathbf{w}}_m)\big\|_{H^1(B_m^+)}\leqslant C\Big[\|\boldsymbol{u}\|_{H^\ell(\Omega)}+\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)}+\|\boldsymbol{g}\|_{H^{\ell-1.5}(\partial\Omega)}\Big]$$

or by the smoothness of $\mathbf{w}$ and θ_m,

$$\big\|\widetilde{\zeta}_m\widetilde{\mathbf{w}}_m\cdot\partial^\ell\widetilde{\boldsymbol{u}}_m\big\|_{H^1(B_m^+)}\leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)}+\|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)}+\|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big].\tag{44}$$

Step 3: Estimate (44) only provides a control on the vector $\widetilde{\zeta}_m \partial^\ell \nabla \widetilde{\boldsymbol{u}}_m$ in the direction of $\mathbf{w}$. Now we proceed to the estimates of the component of $\widetilde{\zeta}_m \partial^\ell \nabla \widetilde{\boldsymbol{u}}_m$ perpendicular to $\mathbf{w}$. Define

$$
\begin{aligned}
\boldsymbol{\varphi}^i &= (-1)^\ell \Big[\widetilde{\zeta}_m \Lambda_\epsilon \partial^{2\ell} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^i_m) - \big(\widetilde{\zeta}_m \widetilde{\mathbf{w}}_m \cdot \Lambda_\epsilon \partial^{2\ell} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big) \frac{\widetilde{\mathbf{w}}^i_m}{|\widetilde{\mathbf{w}}_m|^2} \Big] \circ \theta_m^{-1} \\
&= (-1)^\ell \Big[\widetilde{\zeta}_m \Lambda_\epsilon \partial^{2\ell} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^i_m) - \big(\widetilde{\zeta}_m \Lambda_\epsilon \partial^{2\ell} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^j_m) \big) \frac{\widetilde{\mathbf{w}}^j_m \widetilde{\mathbf{w}}^i_m}{|\widetilde{\mathbf{w}}_m|^2} \Big] \circ \theta_m^{-1} \,.
\end{aligned}
$$

We note that $\boldsymbol{\varphi}$ is the projection of the vector $\widetilde{\zeta}_m \Lambda_\epsilon \partial^{2\ell} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m)$ onto the affine space with normal $\mathbf{w}$, and so $\boldsymbol{\varphi} \in \mathcal{V}$ and can be used as a test function. Using $\boldsymbol{\varphi}$ as a test function in (31), a similar computation suggests that

$$
\begin{aligned}
&\big\| \partial^\ell \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(B^+_m)} + \frac{\lambda}{4} \big\| \partial^\ell \Lambda_\epsilon \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(B^+_m)} \\
&\quad \leqslant C_\delta \Big[\|\boldsymbol{f}\|^2_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|^2_{H^{\ell-1.5}(\partial\Omega)} + \|\boldsymbol{u}\|^2_{H^\ell(\Omega)} \Big] + \delta \big\| \partial^\ell \Lambda_\epsilon \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(B^+_m)} \\
&\qquad + (-1)^{\ell+1} \int_{B^+_m} b^{rs} \widetilde{\boldsymbol{u}}^i_{m,s} \Big[\big(\widetilde{\zeta}_m \Lambda_\epsilon \partial^{2\ell} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^j_m) \big) \frac{\widetilde{\mathbf{w}}^j_m \widetilde{\mathbf{w}}^i_m}{|\widetilde{\mathbf{w}}|^2} \Big]_{,r} dy \\
&\quad \leqslant C_\delta \Big[\|\boldsymbol{f}\|^2_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|^2_{H^{\ell-1.5}(\partial\Omega)} + \|\boldsymbol{u}\|^2_{H^\ell(\Omega)} \Big] + 2\delta \big\| \partial^\ell \Lambda_\epsilon \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(B^+_m)} \\
&\qquad + (-1)^{\ell+1} \int_{B^+_m} \frac{b^{rs}}{|\widetilde{\mathbf{w}}|^2} \widetilde{\zeta}_m \widetilde{\mathbf{w}}_m \cdot \widetilde{\boldsymbol{u}}_{m,s} \big(\Lambda_\epsilon \partial^{2\ell} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^j_{m,r}) \big) \widetilde{\mathbf{w}}^j_m \, dy \\
&\quad \leqslant C_\delta \Big[\|\boldsymbol{f}\|^2_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|^2_{H^{\ell-1.5}(\partial\Omega)} + \|\boldsymbol{u}\|^2_{H^\ell(\Omega)} \Big] + 3\delta \big\| \partial^\ell \Lambda_\epsilon \nabla (\widetilde{\zeta} \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(B^+_m)} \\
&\qquad - \int_{B^+_m} \frac{b^{rs}}{|\widetilde{\mathbf{w}}|^2} \partial^\ell (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_{m,s} \cdot \widetilde{\mathbf{w}}_m) \big(\Lambda_\epsilon \partial^\ell \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^j_{m,r}) \big) \widetilde{\mathbf{w}}^j_m \, dy \,.
\end{aligned}
$$

Applying estimate (44) and Young's inequality,

$$
\begin{aligned}
&- \int_{B^+_m} \frac{b^{rs}}{|\widetilde{\mathbf{w}}|^2} \partial^\ell (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_{m,s} \cdot \widetilde{\mathbf{w}}_m) \big(\Lambda_\epsilon \partial^\ell \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^j_{m,r}) \big) \widetilde{\mathbf{w}}^j_m \, dy \\
&\quad \leqslant C \big\| \partial^\ell (\widetilde{\zeta}_m \widetilde{\mathbf{w}}_m \cdot \nabla \widetilde{\boldsymbol{u}}_m) \big\|_{L^2(\Omega)} \big\| \partial^\ell \Lambda_\epsilon (\widetilde{\zeta}_m \nabla \widetilde{\boldsymbol{u}}_m) \big\|_{L^2(\Omega)} \\
&\quad \leqslant C_\delta \Big[\|\boldsymbol{f}\|^2_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|^2_{H^{\ell-1.5}(\partial\Omega)} + \|\boldsymbol{u}\|^2_{H^\ell(\Omega)} \Big] + \delta \big\| \partial^\ell \Lambda_\epsilon (\widetilde{\zeta}_m \nabla \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(\Omega)} \,;
\end{aligned}
$$

thus by choosing $\delta > 0$ small enough, we conclude that

$$
\| \partial^\ell \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \|^2_{H^1(B^+_m)} \leqslant C \Big[\|\boldsymbol{u}\|^2_{H^\ell(\Omega)} + \|\boldsymbol{f}\|^2_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|^2_{H^{\ell-1.5}(\partial\Omega)} \Big] .
$$

Again, due to the ϵ-independence of the right-hand side, we conclude that

$$
\| \widetilde{\zeta}_m \partial^\ell \widetilde{\boldsymbol{u}}_m \|_{H^1(B^+_m)} \leqslant C \Big[\|\boldsymbol{u}\|_{H^\ell(\Omega)} + \|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-1.5}(\partial\Omega)} \Big] \tag{45}
$$

for some constant $C = C(\|a\|_{\mathscr{C}^\ell(\Omega)}, \|\mathbf{w}\|_{\mathscr{C}^{\ell+1}(\Omega)}, |\partial\Omega|_{\mathscr{C}^{\ell+1}})$.

Step 4: Multiplying (40) by ζ_m and then composing with θ_m, by the Piola identity (27) we obtain that

$$
\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m - \widetilde{\zeta}_m \big(b^{rs} \widetilde{\boldsymbol{u}}_{m,s} \big)_{,r} = \widetilde{\zeta}_m (\boldsymbol{f} \circ \theta_m) \quad \text{a.e. in } B^+_m \,.
$$

Letting $\partial^{\ell-1-j} \nabla^j$ act on the equation above, we find that

$$
\widetilde{\zeta}_m b^{rs} \partial^{\ell-1-j} \nabla^j \widetilde{\boldsymbol{u}}_{m,rs} = \boldsymbol{F}_{(\ell,j)} \quad \text{a.e. in } B^+_m \tag{46}
$$

for some $\boldsymbol{F}_{(\ell,j)} \in L^2(\Omega)$ satisfying

$$\|\boldsymbol{F}_{(\ell,j)}\|_{L^2(\Omega)} \leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{u}\|_{H^\ell(\Omega)}\Big],$$

where the constant C depends on $\|a\|_{\mathscr{C}^\ell(\Omega)}$. By the ellipticity of L, $b^{\rm nn} > 0$; thus (46) further implies that

$$\widetilde{\zeta}_m \partial^{\ell-1-j}\nabla^j \widetilde{\boldsymbol{u}}_{m,\rm nn} = \frac{1}{b^{\rm nn}}\Big[\boldsymbol{F}_{(\ell,j)} - \widetilde{\zeta}_m \sum_{(r,s)\neq(\rm n,n)} b^{rs}\partial^{\ell-1}\nabla^j \widetilde{\boldsymbol{u}}_{m,rs}\Big].$$

As a consequence, with $j=0$ (45) suggests that

$$\big\|\widetilde{\zeta}_m \partial^{\ell-1}\widetilde{\boldsymbol{u}}_{m,\rm nn}\big\|_{L^2(B_m^+)} \leqslant C\Big[\|\boldsymbol{u}\|_{H^\ell(\Omega)} + \|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-1.5}(\partial\Omega)}\Big]$$

which, combined with (45), provides the estimate

$$\big\|\widetilde{\zeta}_m \partial^{\ell-1}\nabla^2\widetilde{\boldsymbol{u}}_m\big\|_{L^2(B_m^+)} \leqslant C\Big[\|\boldsymbol{u}\|_{H^\ell(\Omega)} + \|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-1.5}(\partial\Omega)}\Big].$$

Repeating this process for $j = 1, \cdots, \ell$, we conclude that

$$\|\widetilde{\zeta}_m \nabla^{\ell+1}\widetilde{\boldsymbol{u}}_m\|_{L^2(B_m^+)} \leqslant C\Big[\|\boldsymbol{u}\|_{H^\ell(\Omega)} + \|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-1.5}(\partial\Omega)}\Big], \tag{47}$$

and the combination of (39) and (47), as well as the induction process, proves the theorem. □

3.2. **The case that the coefficients a^{jk} are of Sobolev class.** We are now in the position of studying the regularity of solution $\boldsymbol{u}$ to (28) when the coefficient a^{jk} and the domain Ω is of Sobolev class. We first prove the following rather technical

Theorem 3.5. *Let $\Omega \subseteq \mathbb{R}^{\rm n}$ be a bounded smooth domain. Suppose that for some integer $\rm k > \frac{n}{2}$ and $1 \leqslant \ell \leqslant \rm k$, $a^{jk} \in H^{\rm k}(\Omega)$ satisfies the ellipticity condition*

$$a^{jk}\xi_j\xi_k \geqslant \lambda|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^{\rm n},$$

and $\mathbf{w} \in H^{\max\{\rm k,\ell+1\}}(\Omega)$ (or $\mathbf{w} \in H^{\max\{\rm k-\frac12,\ell+\frac12\}}(\partial\Omega)$) such that $\mathbf{w}$ vanishes nowhere on $\partial\Omega$. Then for all $\boldsymbol{f} \in H^{\ell-1}(\Omega)$ and $g \in H^{\ell-0.5}(\partial\Omega)$, the weak solution $\boldsymbol{u}$ to (28) belongs to $H^{\ell+1}(\Omega)$, and satisfies

$$\|\boldsymbol{u}\|_{H^{\ell+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)} + \mathcal{P}\big(\|a\|_{H^{\rm k}(\Omega)}\big)\Big(\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{H^{-0.5}(\partial\Omega)}\Big)\Big] \tag{48}$$

for some constant $C = C\big(\|\mathbf{w}\|_{H^{\max\{\rm k,\ell+1\}}(\Omega)}\big)$ and some polynomial $\mathcal{P}$.

Proof. Let $\rm E : H^{k+1}(\Omega) \to H^{k+1}(\mathbb{R}^n)$ be an extension operator, and define $a_\epsilon = \eta_\epsilon * ({\rm E}a)$, $\boldsymbol{f}_\epsilon = \eta_\epsilon * ({\rm E}\boldsymbol{f})$, $\mathbf{w}_\epsilon = \eta_\epsilon * ({\rm E}\mathbf{w})$, and g_ϵ be a smooth regularization of $\boldsymbol{g}$ defined by

$$\boldsymbol{g}_\epsilon = \sum_{m=1}^K \sqrt{\zeta_m}\left[\Lambda_\epsilon\big((\sqrt{\zeta_m}\,\boldsymbol{g})\circ\theta_m\big)\right]\circ\theta_m^{-1},$$

where $\{\zeta_m\}_{m=1}^K$ is a collection of smooth cut-off functions, and $\{\theta_m\}_{m=1}^K$ is smooth coordinate charts subordinate to ${\rm spt}(\zeta_m)$, as in the proof of Theorem 3.4. Since a^{jk} satisfies the ellipticity condition

$$a^{jk}\xi_j\xi_k \geqslant \lambda|\xi|^2 \text{ for some } \lambda > 0,$$

and $H^{\rm k}(\Omega) \hookrightarrow C(\Omega)$ (which implies that $a^{ij} \in C(\Omega)$), we find that for all $\varepsilon \ll 1$,

$$a_\epsilon^{jk}(x)\xi_j\xi_k \geqslant \frac{\lambda}{2}|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^{\rm n}, x \in \Omega. \tag{49}$$

Moreover, by the properties of convolution, as $\varepsilon \to 0$, we have

$$a_\epsilon^{jk} \to a^{jk} \quad \text{in} \quad H^{\mathrm{k}}(\Omega)\,, \tag{50a}$$

$$\mathbf{w}_\epsilon \to \mathbf{w} \quad \text{in} \quad H^{\max\{k,\ell+1\}}(\Omega)\,, \tag{50b}$$

$$\boldsymbol{f}_\epsilon \to \boldsymbol{f} \quad \text{in} \quad H^{\ell-1}(\Omega)\,, \tag{50c}$$

$$\boldsymbol{g}_\epsilon \to \boldsymbol{g} \quad \text{in} \quad H^{\ell-0.5}(\partial\Omega)\,. \tag{50d}$$

As a consequence, Theorem 3.2 suggests that there exists a unique weak solution $\boldsymbol{u}^\epsilon$ to

$$\boldsymbol{u}^\epsilon - \frac{\partial}{\partial x_j}\Big(a_\epsilon^{jk}\frac{\partial \boldsymbol{u}^\epsilon}{\partial x_k}\Big) = \boldsymbol{f}_\epsilon \quad \text{in} \quad \Omega\,, \tag{51a}$$

$$\boldsymbol{u}^\epsilon \cdot \mathbf{w}_\epsilon = 0 \quad \text{on} \quad \partial\Omega\,, \tag{51b}$$

$$\mathrm{P}_{\mathbf{w}_\epsilon^\perp}\Big(a_\epsilon^{jk}\frac{\partial \boldsymbol{u}^\epsilon}{\partial x_k}\mathbf{N}_j - \boldsymbol{g}_\epsilon\Big) = \mathbf{0} \quad \text{on} \quad \partial\Omega\,, \tag{51c}$$

and Theorem 3.4 further suggests that $\boldsymbol{u}^\epsilon \in H^s(\Omega)$ for all $s > 0$ which ensures that $\boldsymbol{u}^\epsilon$ is a classical solution to (51). We would like to establish an ε-independent upper bound for $\|\boldsymbol{u}^\epsilon\|_{H^{\ell+1}(\Omega)}$.

Step 1: Similar to Step 2 in the proof of Theorem 3.4, define

$$\boldsymbol{\varphi}^i = (-1)^\ell\big[\widetilde{\zeta}_m \widetilde{\mathbf{w}}_m^i \partial^{2\ell}(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)\big] \circ \theta_m^{-1}\,,$$

in which $\widetilde{\boldsymbol{u}}_m = \boldsymbol{u}^\epsilon \circ \theta_m$ and $\widetilde{\mathbf{w}}_m = \mathbf{w}_\epsilon \circ \theta_m$. The use of $\boldsymbol{\varphi}$ as a test function implies that

$$(\boldsymbol{u}^\epsilon, \boldsymbol{\varphi})_{L^2(\Omega)} + \int_\Omega a_\epsilon^{jk}\frac{\partial \boldsymbol{u}^{\varepsilon i}}{\partial x_k}\frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}dx \leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)}\Big]\big\|\partial^\ell(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)\big\|_{H^1(B_m^+)}\,. \tag{52}$$

As in the proof of Theorem 3.4, we have

$$(\boldsymbol{u}^\epsilon, \boldsymbol{\varphi})_{L^2(\Omega)} = \big\|\partial^\ell \Lambda_\epsilon(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)\big\|^2_{L^2(B_m^+)}\,.$$

Now we focus on the second term of the left-hand side of (52). Integrating by parts in y_α, letting $b_\epsilon^{rs} = (a_\epsilon^{jk} \circ \theta_m)A_k^s A_j^r$ we obtain that

$$\begin{aligned}
\int_\Omega a_\epsilon^{jk}\frac{\partial \boldsymbol{u}^{\varepsilon i}}{\partial x_k}\frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\,dx &= (-1)^\ell \int_{B_m^+} b_\epsilon^{rs}\widetilde{\boldsymbol{u}}^i_{m,s}\big[\widetilde{\zeta}_m \widetilde{\mathbf{w}}_m^i \partial^{2\ell}(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)\big],_r\,dy \\
&= \int_{B_m^+} \partial^\ell\big[b_\epsilon^{rs}(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_s\big]\partial^\ell(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_r\,dy \\
&\quad - \int_{B_m^+} \partial^\ell\big[b_\epsilon^{rs}\widetilde{\boldsymbol{u}}_m^i(\widetilde{\zeta}_m \widetilde{\mathbf{w}}_m^i),_s\big]\partial^\ell(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_r\,dy \\
&\quad - \int_{B_m^+} \partial^{\ell-1}\big[b_\epsilon^{rs}v^i_{m,s}(\widetilde{\zeta}_m \widetilde{\mathbf{w}}_m^i),_r\big]\partial^{\ell+1}(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)\big]dy\,.
\end{aligned} \tag{53}$$

For the first term on the right-hand side of (53), we make use of ellipticity and Young's inequality to obtain that

$$\begin{aligned}
&\int_{B_m^+} \partial^\ell\big[b_\epsilon^{rs}(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_s\big]\partial^\ell(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_r\,dy \\
&\quad = \int_{B_m^+} b_\epsilon^{rs}\partial^\ell(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_s\,\partial^\ell(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_r\,dy \\
&\qquad + \int_{B_m^+} \big[[\![\partial^\ell, b_\epsilon^{rs}]\!](\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_s\big]\partial^\ell(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_r\,dy \\
&\quad \geqslant \big(\frac{\lambda}{8} - \delta\big)\big\|\partial^\ell \nabla(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)\big\|^2_{L^2(\Omega)} - C_\delta\big\|[\![\partial^\ell, b_\epsilon]\!]\nabla(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m)\big\|^2_{L^2(B_m^+)}\,;
\end{aligned}$$

hence, Lemma 2.11 and (21) (with $\epsilon = \dfrac{1}{8}$) imply that

$$\int_{B_m^+} \partial^\ell \big[b_\epsilon^{rs} (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_s \big] \partial^\ell (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_r \, dy \geqslant \big(\frac{\lambda}{8} - \delta\big) \big\| \partial^\ell \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m) \big\|^2_{L^2(\Omega)} - C_\delta \|a\|^2_{H^{\mathrm{k}}(\Omega)} \|\boldsymbol{u}^\epsilon\|^2_{H^{\ell+\frac{7}{8}}(\Omega)} .$$

On the other hand, by (18), we find that

$$\int_{B_m^+} \partial^\ell \big[b_\epsilon^{rs} \widetilde{\boldsymbol{u}}^i_m (\widetilde{\zeta}_m \widetilde{\mathbf{w}}^i_m),_s \big] \partial^\ell (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m),_r \, dy + \int_{B_m^+} \partial^{\ell-1} \big[b_\epsilon^{rs} v^i_{m,s} (\widetilde{\zeta}_m \widetilde{\mathbf{w}}^i_m),_r \big] \partial^{\ell+1} (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m) \, dy \leqslant C_\delta \|a\|^2_{H^{\mathrm{k}}(\Omega)} \|\boldsymbol{u}^\epsilon\|^2_{H^\ell(\Omega)} + \delta \big\| \partial^\ell \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^i_m \widetilde{\mathbf{w}}^i_m) \big\|^2_{L^2(B_m^+)} .$$

As a consequence, by choosing $\delta > 0$ small enough we conclude that

$$\big\| \partial^\ell (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m) \big\|_{L^2(B_m^+)} + \big\| \partial^\ell \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m) \big\|_{L^2(B_m^+)} \leqslant C \Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)} + \|a\|_{H^{\mathrm{k}}(\Omega)} \|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega)} \Big] . \tag{54}$$

Step 2: Similar to Step 3 in the proof of Theorem 3.4, the use of

$$\boldsymbol{\varphi}^i = \Big[\widetilde{\zeta}_m \partial^{2\ell} (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^i_m) - \big(\widetilde{\zeta}_m \widetilde{\mathbf{w}}_m \cdot \partial^{2\ell} (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big) \frac{\widetilde{\mathbf{w}}^i_m}{|\widetilde{\mathbf{w}}_m|^2} \Big] \circ \theta_m^{-1}$$

as a test function implies that

$$\big\| \partial^\ell (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(B_m^+)} + \big(\frac{\lambda}{8} - \delta\big) \big\| \partial^\ell \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(B_m^+)} \leqslant C \Big[\|\boldsymbol{f}\|^2_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|^2_{H^{\ell-0.5}(\partial\Omega)} \Big] + C_\delta \|a\|^2_{H^{\mathrm{k}}(\mathrm{O})} \|\boldsymbol{u}^\epsilon\|^2_{H^{\ell+\frac{7}{8}}(\Omega)} + (-1)^\ell \int_{B_m^+} \frac{\widetilde{\mathbf{w}}^j_m}{|\widetilde{\mathbf{w}}_m|^2} (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m) \partial^{2\ell} (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^j_m) \, dy + (-1)^{\ell+1} \int_{B_m^+} b_\epsilon^{rs} \widetilde{\boldsymbol{u}}^i_{m,s} \Big[\widetilde{\zeta}_m \partial^{2\ell} (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^j_m) \frac{\widetilde{\mathbf{w}}^j_m \widetilde{\mathbf{w}}^i_m}{|\widetilde{\mathbf{w}}_m|^2} \Big],_r \, dy.$$

Integrating by parts in y_α, by Lemma 2.11 and (54) we find that

$$(-1)^\ell \int_{B_m^+} \frac{\widetilde{\mathbf{w}}^j_m}{|\widetilde{\mathbf{w}}_m|^2} (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m) \partial^{2\ell} (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^j_m) \, dy \leqslant C_\delta \big\| \partial^\ell (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m) \big\|^2_{L^2(B_m^+)} + \delta \big\| \partial^\ell (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(B_m^+)} \leqslant C_\delta \Big[\|\boldsymbol{f}\|^2_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|^2_{H^{\ell-0.5}(\partial\Omega)} + \|a\|^2_{H^{\mathrm{k}}(\Omega)} \|\boldsymbol{u}^\epsilon\|^2_{H^{\ell+\frac{7}{8}}(\Omega)} \Big] + \delta \big\| \partial^\ell (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(B_m^+)}$$

for some constant C_δ depending on $\|\mathbf{w}\|_{H^{\max\{\mathrm{k},\ell+1\}}(\Omega)}$, and

$$(-1)^{\ell+1} \int_{B_m^+} b_\epsilon^{rs} \widetilde{\boldsymbol{u}}^i_{m,s} \Big[\widetilde{\zeta}_m \partial^{2\ell} (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^j_m) \frac{\widetilde{\mathbf{w}}^j_m \widetilde{\mathbf{w}}^i_m}{|\widetilde{\mathbf{w}}_m|^2} \Big],_r \, dy \leqslant C \Big[\big\| \partial^\ell (b_\epsilon \nabla \widetilde{\boldsymbol{u}}_m) \big\|_{L^2(B_m^+)} + \big\| b_\epsilon \partial^\ell \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m) \big\|_{L^2(B_m^+)} + \big\| [\![\partial^\ell, b_\epsilon]\!] \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m \cdot \widetilde{\mathbf{w}}_m) \big\|_{L^2(B_m^+)} \Big] \big\| \partial^\ell \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big\|_{L^2(\Omega)} \leqslant C_\delta \|a\|^2_{H^{\mathrm{k}}(\Omega)} \|\boldsymbol{u}^\epsilon\|^2_{H^{\ell+\frac{7}{8}}(\Omega)} + \delta \big\| \partial^\ell \nabla (\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m) \big\|^2_{L^2(\Omega)}$$

in which the constant C_δ also depends on $\|\mathbf{w}\|_{H^{\max\{\mathrm{k},\ell+1\}}(\Omega)}$. Therefore, choosing $\delta > 0$ small enough, we conclude that

$$\begin{aligned}&\big\|\widetilde{\zeta}_m \partial^\ell \widetilde{\boldsymbol{u}}^i_m\big\|_{L^2(B^+_m)} + \big\|\widetilde{\zeta}_m \partial^\ell \nabla \widetilde{\boldsymbol{u}}^i_m\big\|_{L^2(B^+_m)} \\ &\qquad\leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)} + \|a\|_{H^{\mathrm{k}}(\Omega)}\|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega)}\Big]\end{aligned} \tag{55}$$

for some constant $C = C\big(\|\mathbf{w}\|_{H^{\max\{\mathrm{k},\ell+1\}}(\Omega)}\big)$.

Step 3: In this step, we follow the procedure of Step 4 in the proof of Theorem 3.4. Since

$$\boldsymbol{u}^\epsilon - \frac{\partial}{\partial x_j}\Big(a^{jk}_\epsilon \frac{\partial\, \boldsymbol{u}^\epsilon}{\partial x_k}\Big) = \boldsymbol{f}_\epsilon \quad \text{in} \quad \Omega\,,$$

by the Piola identity (27) we find that

$$\widetilde{\zeta}_m \big(b^{rs}_\epsilon \widetilde{\boldsymbol{u}}_{m,s}\big)_{,r} = \widetilde{\zeta}_m\big(\widetilde{\boldsymbol{u}}_m - (\boldsymbol{f}_\epsilon \circ \theta_m)\big) \quad \text{in} \quad B^+_m$$

which, after rearranging terms, implies that

$$\begin{aligned}\widetilde{\zeta}_m b^{\mathrm{nn}}_\epsilon \widetilde{\boldsymbol{u}}_{m,\mathrm{nn}} = \widetilde{\zeta}_m\Big[&\widetilde{\boldsymbol{u}}_m - (\boldsymbol{f}_\epsilon\circ\theta_m) - b^{\mathrm{nn}}_{\varepsilon,\mathrm{n}}\widetilde{\boldsymbol{u}}_{m,\mathrm{n}} - \sum_{(r,s)\neq(\mathrm{n},\mathrm{n})} b^{rs}_{\varepsilon,r}\widetilde{\boldsymbol{u}}_{m,s} \\ &- \sum_{(r,s)\neq(\mathrm{n},\mathrm{n})} b^{rs}_\epsilon \widetilde{\boldsymbol{u}}_{m,sr}\Big] \qquad \text{in} \quad B^+_m\,.\end{aligned} \tag{56}$$

First, it is easy to see that

$$\big\|\partial^{\ell-1-j}\nabla^j(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m)\big\|_{L^2(B^+_m)} + \big\|\partial^{\ell-1-j}\nabla^j\big[\widetilde{\zeta}_m(\boldsymbol{f}_\epsilon\circ\theta_m)\big]\big\|_{L^2(B^+_m)} \leqslant C\Big[\|\boldsymbol{u}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)}\Big]\,.$$

Moreover, since $\ell+1 \leqslant \mathrm{k}$, by Proposition 2.4 (with $\epsilon = \dfrac{1}{8}$) we find that

$$\begin{aligned}&\big\|\partial^{\ell-1-j}\nabla^j(\widetilde{\zeta}_m b^{\mathrm{nn}}_{,\mathrm{n}}\widetilde{\boldsymbol{u}}_{m,\mathrm{n}})\big\|_{L^2(B^+_m)} + \sum_{(r,s)\neq(\mathrm{n},\mathrm{n})} \big\|\partial^{\ell-1-j}\nabla^j b^{rs}_{\varepsilon,r}\widetilde{\boldsymbol{u}}_{m,s}\big\|_{L^2(B^+_m)} \\ &\qquad\leqslant C\sum_{r=0}^{\ell-1}\|D^{\ell-r}aD^{r+1}\boldsymbol{u}^\epsilon\|_{L^2(\Omega)} \\ &\qquad\leqslant C\sum_{r=1}^{\ell}\|D^{\ell+1-r}aD^r\boldsymbol{u}^\epsilon\|_{L^2(\Omega)} \leqslant C_\epsilon\|a\|_{H^{\mathrm{k}}(\Omega)}\|\boldsymbol{u}\|_{H^{\ell+\frac{7}{8}}(\Omega)}\,.\end{aligned}$$

Finally, by Lemma 2.11 (with $\epsilon = \dfrac{1}{8}$ again),

$$\begin{aligned}\big\|[\![\partial^{\ell-1-j}\nabla^j, \widetilde{\zeta}_m b^{\mathrm{nn}}_\epsilon]\!]\widetilde{\boldsymbol{u}}_{m,\mathrm{nn}}\big\|_{L^2(B^+_m)} &+ \sum_{(r,s)\neq(\mathrm{n},\mathrm{n})}\big\|[\![\partial^{\ell-1-j}\nabla^j, \widetilde{\zeta}_m b^{rs}_\epsilon]\!]\widetilde{\boldsymbol{u}}_{m,rs}\big\|_{L^2(B^+_m)} \\ &\leqslant C_\epsilon\|a\|_{H^{\mathrm{k}}(\Omega)}\|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega)}\,.\end{aligned}$$

Therefore, letting $\partial^{\ell-1-j}\nabla^j$ act on (56), we obtain that

$$\widetilde{\zeta}_m b^{\mathrm{nn}}_\epsilon \partial^{\ell-1-j}\nabla^j\widetilde{\boldsymbol{u}}_{m,\mathrm{nn}} = G_{(\ell,j)} - \sum_{(r,s)\neq(\mathrm{n},\mathrm{n})}\widetilde{\zeta}_m b^{rs}_\epsilon \partial^{\ell-1-j}\nabla^j\widetilde{\boldsymbol{u}}_{m,rs} \tag{57}$$

for some $\boldsymbol{G}_{(\ell,j)}$ satisfying

$$\|\boldsymbol{G}_{(\ell,j)}\|_{L^2(B^+_m)} \leqslant C\Big[\|\boldsymbol{u}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|a\|_{H^{\mathrm{k}}(\Omega)}\|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega)}\Big]\,.$$

Now we argue by induction on j. By the ellipticity condition (49), we find that $b_\epsilon^{\rm nn} \geqslant \dfrac{\lambda}{4}$. As a consequence, with $j=0$, (55) and (57) suggest that

$$\big\|\widetilde{\zeta}_m \partial^{\ell-1} \widetilde{\boldsymbol{u}}_{m,\rm nn}\big\|_{L^2(B_m^+)} \leqslant \|\boldsymbol{G}_{(\ell,j)}\|_{L^2(B_m^+)} + \sum_{(r,s)\neq(\rm n,n)} \|b_\epsilon^{rs}\|_{L^\infty(B_m^+)} \big\|\widetilde{\zeta}_m \partial^{\ell-1} \widetilde{\boldsymbol{u}}_{m,rs}\big\|_{L^2(B_m^+)}$$
$$\leqslant C\Big[\|\boldsymbol{u}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-1.5}(\partial\Omega)} + \|a\|_{H^{\rm k}(\Omega)}\|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega)}\Big]$$

which, combined with (55), provides the estimate

$$\big\|\widetilde{\zeta}_m \partial^{\ell-1}\nabla^2 \widetilde{\boldsymbol{u}}_m\big\|_{L^2(B_m^+)} \leqslant C\Big[\|\boldsymbol{u}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-1.5}(\partial\Omega)} + \|a\|_{H^{\rm k}(\Omega)}\|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega)}\Big].$$

Repeating this process for $j=1,\cdots,\ell$, we conclude that

$$\begin{aligned}&\big\|\widetilde{\zeta}_m \nabla^\ell \widetilde{\boldsymbol{u}}^i_m\big\|_{L^2(B_m^+)} + \big\|\widetilde{\zeta}_m \nabla^{\ell+1} \widetilde{\boldsymbol{u}}^i_m\big\|_{L^2(B_m^+)} \\ &\qquad \leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)} + \|a\|_{H^{\rm k}(\Omega)}\|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega)}\Big]\end{aligned} \tag{58}$$

for some constant $C = C\big(\|\mathbf{w}\|_{H^{\max\{\rm k,\ell+1\}}(\Omega)}\big)$.

Step 4: Let $\chi \geqslant 0$ be a smooth cut-off function so that $\mathrm{spt}(\chi) \subset\subset \Omega$. Then the same computation as in the previous steps also implies that

$$\|\chi\nabla^\ell \boldsymbol{u}^\epsilon\|_{L^2(\Omega)} + \|\chi\nabla^{\ell+1}\boldsymbol{u}^\epsilon\|_{L^2(\Omega)} \leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|a\|_{H^{\rm k}(\Omega)}\|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega)}\Big]. \tag{59}$$

The combination of (58) and (59) then suggests that

$$\|\boldsymbol{u}^\epsilon\|_{H^{\ell+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)} + \|a\|_{H^{\rm k}(\Omega)}\|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega)}\Big] \tag{60}$$

for some constant $C = C\big(\|\mathbf{w}\|_{H^{\max\{\rm k,\ell+1\}}(\Omega)}\big)$. By interpolation,

$$\|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega)} \leqslant C\|\boldsymbol{u}^\epsilon\|^{1-\frac{1}{8\ell}}_{H^{\ell+1}(\Omega)}\|\boldsymbol{u}^\epsilon\|^{\frac{1}{8\ell}}_{H^1(\Omega)};$$

thus Young's inequality suggests that

$$\|\boldsymbol{u}^\epsilon\|_{H^{\ell+1}(\Omega)} \leqslant C_\delta\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)} + \mathcal{P}\big(\|a\|_{H^{\rm k}(\Omega)}\big)\|\boldsymbol{u}^\epsilon\|_{H^1(\Omega)}\Big] + \delta\|\boldsymbol{u}^\epsilon\|_{H^{\ell+1}(\Omega)}$$

for some polynomial $\mathcal{P}$. Finally, the inequality (33) is established by choosing $\delta>0$ small enough and then letting $\varepsilon\to 0$, and using the H^1-estimate (32). ▫

Corollary 3.6. *Let $\Omega \subseteq \mathbb{R}^{\rm n}$ be a bounded $H^{\rm k+1}$-domain for some integer $\rm k > \dfrac{n}{2}$. Suppose that $a^{jk} \in H^{\rm k}(\Omega)$ satisfies the ellipticity condition*

$$a^{jk}\xi_j\xi_k \geqslant \lambda|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^{\rm n},$$

and for some $1 \leqslant \ell \leqslant \rm k$, $\mathbf{w} \in H^{\max\{\rm k,\ell+1\}}(\Omega)$ (or $\mathbf{w} \in H^{\max\{\rm k-\frac{1}{2},\ell+\frac{1}{2}\}}(\partial\Omega)$) such that $\mathbf{w}$ vanishes nowhere on $\partial\Omega$. Then for all $\boldsymbol{f} \in H^{\ell-1}(\Omega)$ and $\boldsymbol{g} \in H^{\ell-0.5}(\partial\Omega)$, the weak solution $\boldsymbol{u}$ to (28) belongs to $H^{\ell+1}(\Omega)$, and satisfies

$$\|\boldsymbol{u}\|_{H^{\ell+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)} + \mathcal{P}\big(\|a\|_{H^{\rm k}(\Omega)}\big)\Big(\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{H^{-0.5}(\partial\Omega)}\Big)\Big] \tag{61}$$

for some constant $C = C\big(\|\mathbf{w}\|_{H^{\max\{\rm k,\ell+1\}}(\Omega)}, |\partial\Omega|_{H^{\rm k+0.5}}\big)$ and some polynomial $\mathcal{P}$.

Proof. Let $\psi : \mathrm{O} \to \Omega$ be an $H^{\mathrm{k}+1}$-diffeomorphism. Making a change of variable $x = \psi(y)$, with A denoting $(\nabla\psi)^{-1}$ we can rewrite (28) as

$$\begin{aligned}
\bar{\boldsymbol{u}} - \frac{\partial}{\partial y_r}\Big(\bar{a}^{jk}\mathrm{A}^r_j\mathrm{A}^s_k\frac{\partial \bar{\boldsymbol{u}}}{\partial y_s}\Big) &= \bar{\boldsymbol{f}} + \bar{a}^{jk}\mathrm{A}^s_k\frac{\partial \mathrm{A}^r_j}{\partial y_r}\frac{\partial \bar{\boldsymbol{u}}}{\partial y_s} && \text{in} \quad \mathrm{O}\,,\\
\bar{\boldsymbol{u}}\cdot\bar{\mathbf{w}} &= 0 && \text{on} \quad \partial\mathrm{O}\,,\\
\mathrm{P}_{\bar{\mathbf{w}}^\perp}\Big(\bar{a}^{jk}\mathrm{A}^r_j\mathrm{A}^s_k\frac{\partial \bar{\boldsymbol{u}}}{\partial y_s}\bar{\mathbf{N}}_r - \bar{\boldsymbol{g}}\Big) &= \mathbf{0} && \text{on} \quad \partial\mathrm{O}\,,
\end{aligned}$$

where we use the bar notation to denote the variable defined on O through the composition with ψ; that is,

$$\bar{a} = a\circ\psi\,,\quad \bar{\boldsymbol{u}} = \boldsymbol{u}\circ\psi\,,\quad \bar{\mathbf{w}} = \mathbf{w}\circ\psi\,,\quad \bar{\boldsymbol{f}} = \boldsymbol{f}\circ\psi\,,\quad \bar{\boldsymbol{g}} = \boldsymbol{g}\circ\psi\,,$$

and $\bar{\mathbf{N}}$ is the outward-pointing unit normal to O. By Proposition 2.4, Corollary 2.6, and Proposition 2.9, we find that

$$\begin{aligned}
\|\bar{a}^{jk}\mathrm{A}^s_k\mathrm{A}^r_j\|_{H^{\mathrm{k}}(\mathrm{O})} &\leqslant C(|\partial\Omega|_{H^{k+0.5}})\|a\|_{H^{\mathrm{k}}(\Omega)}\,,\\
\|\bar{\mathbf{w}}\|_{H^{\max\{\mathrm{k},\ell+1\}}(\Omega)} &\leqslant C(|\partial\Omega|_{H^{k+0.5}})\|\mathbf{w}\|_{H^{\max\{\mathrm{k},\ell+1\}}(\Omega)}\,,\\
\|\bar{\boldsymbol{f}}\|_{H^{\ell-1}(\mathrm{O})} + \|\bar{\boldsymbol{g}}\|_{H^{\ell-0.5}(\partial\mathrm{O})} &\leqslant C(|\partial\Omega|_{H^{k+0.5}})\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)}\Big]\,;
\end{aligned}$$

thus Theorem 3.5 implies that

$$\begin{aligned}
\|\bar{\boldsymbol{u}}\|_{H^{\ell+1}(\mathrm{O})} &\leqslant C\Big[\|\bar{\boldsymbol{f}}\|_{H^{\ell-1}(\mathrm{O})} + \|\bar{\boldsymbol{g}}\|_{H^{\ell-0.5}(\partial\mathrm{O})} + \mathcal{P}\big(\|\mathrm{A}\bar{a}\mathrm{A}^{\mathrm{T}}\|_{H^{\mathrm{k}}(\mathrm{O})}\big)\Big(\|\bar{\boldsymbol{f}}\|_{L^2(\mathrm{O})} + \|\bar{\boldsymbol{g}}\|_{H^{-0.5}(\partial\mathrm{O})}\Big)\Big]\\
&\leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\mathrm{O})} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)} + \mathcal{P}\big(\|a\|_{H^{\mathrm{k}}(\Omega)}, |\partial\Omega|_{H^{\mathrm{k}+0.5}}\big)\Big(\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{H^{-0.5}(\partial\Omega)}\Big)\Big]
\end{aligned}$$

for some constant $C = C\big(\|\mathbf{w}\|_{H^{\max\{\mathrm{k},\ell+1\}}(\Omega)}, |\partial\Omega|_{H^{k+0.5}}\big)$. Estimate (61) then follows from Proposition 2.9. ▫

Corollary 3.7. *Let $\Omega \subseteq \mathbb{R}^{\mathrm{n}}$ be a bounded $H^{\mathrm{k}+1}$-domain for some integer $\mathrm{k} > \frac{\mathrm{n}}{2}$, and $a^{jk} \in H^{\mathrm{k}}(\Omega)$ satisfies the ellipticity condition*

$$a^{jk}\xi_j\xi_k \geqslant \lambda|\xi|^2 \qquad \forall\,\xi\in\mathbb{R}^{\mathrm{n}}\,.$$

Let ℓ be an integer such that $1 \leqslant \ell \leqslant \mathrm{k}$. Then

1. *For any $f \in H^{\ell-1}(\Omega)$, the weak solution $u \in H^1_0(\Omega)$ to the Dirichlet problem*

$$\begin{aligned}
-\frac{\partial}{\partial x_j}\Big(a^{jk}\frac{\partial u}{\partial x_k}\Big) &= f && \text{in} \quad \Omega\,,\\
u &= 0 && \text{on} \quad \partial\Omega\,,
\end{aligned}$$

   belongs to $H^{\ell+1}(\Omega)$, and satisfies

$$\|u\|_{H^{\ell+1}(\Omega)} \leqslant C\|f\|_{H^{\ell-1}(\Omega)} \tag{62}$$

   for some constant $C = C\big(\|a\|_{H^{\mathrm{k}}(\Omega)}, |\partial\Omega|_{H^{\mathrm{k}+0.5}}\big)$.

2. *For any $f \in H^{\ell-1}(\Omega)$ and $g \in H^{\ell-0.5}(\partial\Omega)$, the weak solution $v \in H^1(\Omega)$ to the Neumann problem*

$$\begin{aligned}
v - \frac{\partial}{\partial x_j}\Big(a^{jk}\frac{\partial v}{\partial x_k}\Big) &= f && \text{in} \quad \Omega\,,\\
a^{jk}\frac{\partial u}{\partial x_k}\mathbf{N}_j &= g && \text{on} \quad \partial\Omega\,,
\end{aligned}$$

belongs to $H^{\ell+1}(\Omega)$*, and satisfies*

$$\|v\|_{H^{\ell+1}(\Omega)} \leqslant C\Big[\|f\|_{H^{\ell-1}(\Omega)} + \|g\|_{H^{\ell-0.5}(\partial\Omega)}\Big] \tag{63}$$

for some constant $C = C\big(\|a\|_{H^{\mathrm{k}}(\Omega)}, |\partial\Omega|_{H^{\mathrm{k}+0.5}}\big)$.

Proof. It suffices to prove the case that u and v are both scalar functions.

(1) Let $\mathbf{w} = (1, 0, \cdots, 0)$, and $\boldsymbol{u}$ be the solution to

$$\boldsymbol{u} - \frac{\partial}{\partial x_j}\Big(a^{jk}\frac{\partial \boldsymbol{u}}{\partial x_k}\Big) = (f+u, 0, \cdots, 0) \qquad \text{in} \quad \Omega\,, \tag{64}$$

$$\boldsymbol{u}\cdot\mathbf{w} = 0 \qquad \text{on} \quad \partial\Omega\,, \tag{65}$$

$$\mathrm{P}_{\mathbf{w}^\perp}\Big(a^{jk}\frac{\partial \boldsymbol{u}}{\partial x_k}\mathbf{N}_j\Big) = \mathbf{0} \qquad \text{on} \quad \partial\Omega\,. \tag{66}$$

Then $\boldsymbol{u} = (u, 0, \cdots, 0)$ and (61) implies that

$$\|u\|_{H^{\ell+1}(\Omega)} \leqslant C\|f+u\|_{H^{\ell-1}(\Omega)} \leqslant C\Big[\|f\|_{H^{\ell-1}(\Omega)} + \|u\|_{H^{\ell-1}(\Omega)}\Big]$$

for some constant $C = C\big(\|a\|_{H^{\mathrm{k}}(\Omega)}, |\partial\Omega|_{H^{\mathrm{k}+0.5}}\big)$. By interpolation and Young's inequality,

$$\|u\|_{H^{\ell+1}(\Omega)} \leqslant C\|f\|_{H^{\ell-1}(\Omega)} + C_\delta\|u\|_{H^1(\Omega)} + \delta\|u\|_{H^{\ell+1}(\Omega)}\,;$$

thus (62) follows from choosing $\delta > 0$ small enough and the estimate for the weak solution.

(2) Let $\mathbf{w} = (0, 1, 0, \cdots, 0)$, and $\boldsymbol{v}$ be the solution to

$$\boldsymbol{v} - \frac{\partial}{\partial x_j}\Big(a^{jk}\frac{\partial \boldsymbol{v}}{\partial x_k}\Big) = (0, f, 0, \cdots, 0) \qquad \text{in} \quad \Omega\,, \tag{67}$$

$$\boldsymbol{v}\cdot\mathbf{w} = 0 \qquad \text{on} \quad \partial\Omega\,, \tag{68}$$

$$\mathrm{P}_{\mathbf{w}^\perp}\Big(a^{jk}\frac{\partial \boldsymbol{v}}{\partial x_k}\mathbf{N}_j\Big) = (0, g, 0, \cdots, 0) \qquad \text{on} \quad \partial\Omega\,. \tag{69}$$

Then $\boldsymbol{v} = (0, v, 0, \cdots, 0)$ and so (63) follows from (61). □

In general, elliptic estimates with Sobolev class coefficients a^{jk} have a nonlinear dependence on the Sobolev norm of a^{jk}. There are, however, situations when the estimate becomes linear with respect to the Sobolev norm of a^{jk}.

Theorem 3.8. *Suppose that the assumptions of Theorem 3.5 are satisfied with* $\ell = \mathrm{k}$*, and that furthermore*

$$\|a - \mathrm{Id}\|_{L^\infty(\Omega)} \leqslant \epsilon \ll 1\,.$$

Then the solution $\boldsymbol{u} \in H^{\mathrm{k}+1}(\Omega)$ *to* (28) *satisfies*

$$\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{f}\|_{H^{\mathrm{k}-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\mathrm{k}-0.5}(\partial\Omega)} + \big(1 + \|a\|_{H^{\mathrm{k}}(\Omega)}\big)\|\nabla\boldsymbol{u}\|_{L^\infty(\Omega)}\Big] \tag{70}$$

for some constant $C = C\big(\|\mathrm{w}\|_{H^{\mathrm{k}+1}(\Omega)}\big)$*. (Recall that* $\mathbf{w}$ *is an* $H^{\mathrm{k}+1}(\Omega)$ *vector field defined in a neighborhood of* $\partial\Omega$ *which vanishes nowhere on* $\partial\Omega$*.)*

Remark 3.9. *As we noted, inequality (70) is linear with respect to the highest-order norms. This permits the use of linear interpolation to extend this inequality of fractional-order Sobolev spaces.*

Proof. By Theorem 3.5 we know that $\boldsymbol{u} \in H^{k+1}(\Omega)$ so equation (28) holds in the pointwise sense. We rewrite (28) as

$$\begin{aligned} \boldsymbol{u} - \Delta \boldsymbol{u} = \mathbf{f} \equiv \frac{\partial}{\partial x_j}\Big((a^{jk} - \delta^{jk}) \frac{\partial \boldsymbol{u}}{\partial x_k} \Big) + \boldsymbol{f} & \qquad \text{in} \quad \Omega\,, \\ \boldsymbol{u} \cdot \mathbf{N} = 0 & \qquad \text{on} \quad \partial\Omega\,, \\ \mathrm{P}_{\mathbf{w}^\perp}\Big(\frac{\partial \boldsymbol{u}}{\partial \mathbf{N}}\Big) = \mathbf{g} \equiv \mathrm{P}_{\mathbf{w}^\perp}\Big((\delta^{jk} - a^{jk}) \frac{\partial \boldsymbol{u}}{\partial x_k} \mathbf{N}_j + \boldsymbol{g}\Big) & \qquad \text{on} \quad \partial\Omega\,. \end{aligned}$$

We then conclude from Theorem 3.5 that

$$\begin{aligned} \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} &\leqslant C\Big[\|\mathbf{f}\|_{H^{k-1}(\Omega)} + \|\mathbf{g}\|_{H^{k-0.5}(\partial\Omega)} \Big] \\ &\leqslant C\Big[\|\boldsymbol{f}\|_{H^{k-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{k-0.5}(\partial\Omega)} + \Big\| \frac{\partial}{\partial x_j}\Big((\delta^{jk} - a^{jk}) \frac{\partial \boldsymbol{u}}{\partial x_k}\Big) \Big\|_{H^{k-1}(\Omega)} \\ &\qquad + \Big\| \mathrm{P}_{\mathbf{w}^\perp}\Big((\delta^{jk} - a^{jk}) \frac{\partial \boldsymbol{u}}{\partial x_k} \mathbf{N}_j \Big) \Big\|_{H^{k-0.5}(\partial\Omega)} \Big] \end{aligned}$$

for some constant $C = C\big(\|\mathbf{w}\|_{H^{k+1}(\Omega)}\big)$. By Lemma 2.12,

$$\begin{aligned} \Big\| \frac{\partial}{\partial x_j}\Big((\delta^{jk} - a^{jk}) \frac{\partial \boldsymbol{u}}{\partial x_k}\Big) \Big\|_{H^{k-1}(\Omega)} &\leqslant \Big\| (\delta^{jk} - a^{jk}) \frac{\partial \boldsymbol{u}}{\partial x_k} \Big\|_{H^{k}(\Omega)} \\ &\leqslant C\Big[\|\delta - a\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{u}\|_{H^k(\Omega)} + \|\delta - a\|_{H^k(\Omega)} \|\nabla \boldsymbol{u}\|_{L^\infty(\Omega)} \Big] \\ &\leqslant C\epsilon \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + C\big(1 + \|a\|_{H^k(\Omega)}\big) \|\nabla \boldsymbol{u}\|_{L^\infty(\Omega)}\,. \end{aligned}$$

Similarly, by the trace estimate (and the fact that $k - 0.5 > \dfrac{n-1}{2}$, where $n-1$ is the dimension of $\partial\Omega$),

$$\begin{aligned} \Big\| \mathrm{P}_{\mathbf{w}^\perp}\Big((\delta^{jk} - a^{jk}) \frac{\partial \boldsymbol{u}}{\partial x_k} \mathbf{N}_j \Big) \Big\|_{H^{k-0.5}(\partial\Omega)} &\leqslant C \Big\| (\delta^{jk} - a^{jk}) \frac{\partial \boldsymbol{u}}{\partial x_k} \mathbf{N}_j \Big\|_{H^{k-0.5}(\partial\Omega)} \\ &\leqslant C\Big[\|\delta - a\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{u}\|_{H^{k-0.5}(\partial\Omega)} + \|\delta - a\|_{H^{k-0.5}(\partial\Omega)} \|\nabla \boldsymbol{u}\|_{L^\infty(\partial\Omega)} \\ &\leqslant C\epsilon \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + C\big(1 + \|a\|_{H^k(\Omega)}\big) \|\nabla \boldsymbol{u}\|_{L^\infty(\partial\Omega)} \end{aligned}$$

for some constant $C = C\big(\|\mathbf{w}\|_{H^{k+1}(\Omega)}\big)$. Moreover, the embedding $H^{\frac{n}{2}+\delta}(\Omega) \hookrightarrow \mathscr{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$ suggests that $\nabla \boldsymbol{u}$ is uniformly Hölder; thus $\|\nabla \boldsymbol{u}\|_{L^\infty(\partial\Omega)} \leqslant \|\nabla \boldsymbol{u}\|_{L^\infty(\Omega)}$. (70) then follows from the assumption that $\epsilon \ll 1$. □

In the same way that we proved Theorem 3.5, we can prove the following complimentary result:

Theorem 3.10. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded H^{k+1}-domain for some integer $k > \dfrac{n}{2}$. Suppose that $a^{jk} \in H^k(\Omega)$ satisfies the ellipticity condition*

$$a^{jk} \xi_j \xi_k \geqslant \lambda |\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^n\,,$$

and for some $1 \leqslant \ell \leqslant k$, $\mathbf{w} \in H^{\max\{k, \ell+1\}}(\Omega)$ (or $\mathbf{w} \in H^{\max\{k-\frac{1}{2}, \ell+\frac{1}{2}\}}(\partial\Omega)$) such that $\mathbf{w}$ vanishes nowhere on $\partial\Omega$. Then for all $\boldsymbol{f} \in H^{\ell-1}(\Omega)$ and $g \in H^{\ell-0.5}(\partial\Omega)$, there exists a solution $\boldsymbol{u}$ to

$$\boldsymbol{u}^i - \frac{\partial}{\partial x_j}\Big(a^{jk} \frac{\partial \boldsymbol{u}^i}{\partial x_k} \Big) = \boldsymbol{f}^i \qquad \text{in} \quad \Omega\,, \tag{71a}$$

$$\boldsymbol{u} \times \mathbf{w} = \mathbf{0} \qquad \text{on} \quad \partial\Omega\,, \tag{71b}$$

$$a^{jk} \frac{\partial \boldsymbol{u}^i}{\partial x_k} \mathbf{N}_j \mathbf{w}^i = g \qquad \text{on} \quad \partial\Omega\,, \tag{71c}$$

and satisfies

$$\|\boldsymbol{u}\|_{H^{\ell+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\partial\Omega)} + \mathcal{P}\big(\|a\|_{H^{\mathrm{k}}(\Omega)}\big)\Big(\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{H^{-0.5}(\partial\Omega)}\Big)\Big] \tag{72}$$

for some constant $C = C\big(\|\mathbf{w}\|_{H^{\max\{\mathrm{k},\ell+1\}}(\Omega)}, |\partial\Omega|_{H^{k+0.5}}\big)$ *and some polynomial* $\mathcal{P}$*.*

4. The Proof of Theorem 1.2

In this section, we prove our main regularity result given by Theorem 1.2. We first establish need the following

Lemma 4.1. *Let* $\Omega \subseteq \mathbb{R}^3$ *be a bounded* $H^{\mathrm{k}+1}$*-domain with outward-pointing normal* $\mathbf{N}$*, and* $(\mathcal{U}, \varphi)$ *be a chart with* $\theta \equiv \varphi^{-1}$*. Define the metric* $g_{\alpha\beta} = \theta,_\alpha \cdot \theta,_\beta$ *induced by the chart, and* $[g^{\alpha\beta}]$ *be the inverse matrix of* $[g_{\alpha\beta}]$*. Then for every vector field* $\boldsymbol{w} : \Omega \to \mathbb{R}^3$*,*

$$\mathrm{P}_{\mathbf{N}^\perp}\Big(\frac{\partial \boldsymbol{w}}{\partial \mathbf{N}}\Big) \circ \theta = (\mathrm{curl}\boldsymbol{w} \times \mathbf{N}) \circ \theta + g^{\alpha\beta}\theta,_\alpha \big[(\boldsymbol{w}\cdot\mathbf{N})\circ\theta\big],_\beta - g^{\alpha\beta}g^{\gamma\delta}\big[(\boldsymbol{w}\circ\theta)\cdot\theta,_\delta\big]b_{\gamma\beta}\,\theta,_\alpha\ , \tag{73}$$

where $b_{\gamma\delta} = -\theta,_{\gamma\delta}\cdot(\mathbf{N}\circ\theta)$ *denotes the second fundamental form.*

Proof. Define

$$\Theta(y) = \theta(y_1, y_2, 0) + y_3(\mathbf{N}\circ\theta)(y_1, y_2, 0)\,,$$

and $\mathcal{G}_{ij} = \Theta,_i \cdot \Theta,_j$ with inverse $\mathcal{G}^{ij}$. Let $\widetilde{\mathbf{N}} \equiv (\mathbf{N}\circ\theta)\big|_{y_3=0}$, and $\widetilde{\boldsymbol{f}} \equiv \boldsymbol{f}\circ\Theta$ if $\boldsymbol{f} \neq \mathbf{N}$. Since $\Theta,_1$, $\Theta,_2 \perp \widetilde{\mathbf{N}}$, for every vector $v \in \mathbb{R}^3$, $\widetilde{\boldsymbol{v}}$ can be expressed as the linear combination of $\Theta,_1$, $\Theta,_2$ and $\widetilde{\mathbf{N}}$. In particular, we have

$$\widetilde{\boldsymbol{v}}^i = (\widetilde{\boldsymbol{v}}\cdot\widetilde{\mathbf{N}})\widetilde{\mathbf{N}}^i + (\mathcal{G}^{\alpha\beta}\Theta^j,_\beta\,\widetilde{\boldsymbol{v}}^j)\Theta^i,_\alpha \equiv \widetilde{\boldsymbol{v}}_3\widetilde{\mathbf{N}}^i + \widetilde{\boldsymbol{v}}_\alpha\Theta^i,_\alpha$$

and

$$\boldsymbol{f},_k\circ\Theta = \widetilde{\boldsymbol{f}},_3\widetilde{\mathbf{N}}^3 + \mathcal{G}^{\alpha\beta}\widetilde{\boldsymbol{f}},_\beta\,\Theta^k,_\alpha\ .$$

To see (73), we first note that

$$\frac{\partial \boldsymbol{w}^i}{\partial\mathbf{N}}\circ\theta = \big[\widetilde{\boldsymbol{w}}_3\widetilde{\mathbf{N}}^i + \widetilde{\boldsymbol{w}}_\alpha\Theta^i,_\alpha\big],_3\Big|_{y_3=0} = \big[\widetilde{\boldsymbol{w}}_{3,3}\widetilde{\mathbf{N}}^i + \widetilde{\boldsymbol{w}}_{\alpha,3}\Theta^i,_\alpha + \widetilde{\boldsymbol{w}}_\alpha\widetilde{\mathbf{N}}^i,_\alpha\big]\Big|_{y_3=0}; \tag{74}$$

thus, since $\widetilde{\mathbf{N}}\cdot\Theta,_\alpha = \widetilde{\mathbf{N}}\cdot\widetilde{\mathbf{N}},_\alpha = 0$,

$$\begin{aligned}
&\mathrm{P}_{\mathbf{N}^\perp}\Big(\frac{\partial\boldsymbol{w}}{\partial\mathbf{N}}\Big)\circ\theta \\
&\quad= \big[\widetilde{\boldsymbol{w}}_{3,3}\widetilde{\mathbf{N}}^i + \widetilde{\boldsymbol{w}}_{\alpha,3}\Theta^i,_\alpha + \widetilde{\boldsymbol{w}}_\alpha\widetilde{\mathbf{N}}^i,_\alpha\big]\Big|_{y_3=0} - \big[\widetilde{\boldsymbol{w}}_{3,3}\widetilde{\mathbf{N}}^k + \widetilde{\boldsymbol{w}}_{\alpha,3}\Theta^k,_\alpha + \widetilde{\boldsymbol{w}}_\alpha\widetilde{\mathbf{N}}^k,_\alpha\big]\Big|_{y_3=0}\widetilde{\mathbf{N}}^k\widetilde{\mathbf{N}}^i \\
&\quad= \widetilde{\boldsymbol{w}}_{\alpha,3}\theta^i,_\alpha + \widetilde{\boldsymbol{w}}_\alpha\widetilde{\mathbf{N}}^i,_\alpha\ .
\end{aligned} \tag{75}$$

Moreover, by the identity

$$(\mathrm{curl}\boldsymbol{w}\times\mathbf{N})^i = \varepsilon_{ijk}\varepsilon_{jrs}\boldsymbol{w}^s,_r\mathbf{N}^k = (\delta_{is}\delta_{kr} - \delta_{ir}\delta_{ks})\boldsymbol{w}^s,_r\mathbf{N}^k = (\boldsymbol{w}^i,_k - \boldsymbol{w}^k,_i)\mathbf{N}^k\,,$$

we find that

$$\begin{aligned}
(\mathrm{curl}\boldsymbol{w}\times\mathbf{N})^i\circ\theta &= \big[(\widetilde{\boldsymbol{w}}^i,_3\widetilde{\mathbf{N}}^k + \mathcal{G}^{\alpha\beta}\widetilde{\boldsymbol{w}}^i,_\beta\Theta^k,_\alpha - \widetilde{\boldsymbol{w}}^k,_3\widetilde{\mathbf{N}}^i - \mathcal{G}^{\alpha\beta}\widetilde{\boldsymbol{w}}^k,_\beta\Theta^i,_\alpha)\widetilde{\mathbf{N}}^k\big]\Big|_{y_3=0} \\
&= \big[\widetilde{\boldsymbol{w}}^i,_3 - \widetilde{\mathbf{N}}^i\widetilde{\boldsymbol{w}}^k,_3\widetilde{\mathbf{N}}^k - g^{\alpha\beta}\Theta^i,_\alpha\,\widetilde{\boldsymbol{w}}^k,_\beta\widetilde{\mathbf{N}}^k\big]\Big|_{y_3=0} \\
&= \big[(\widetilde{\boldsymbol{w}}_3\widetilde{\mathbf{N}}^i + \widetilde{\boldsymbol{w}}_\alpha\Theta^i,_\alpha),_3 - \widetilde{\mathbf{N}}^i(\widetilde{\boldsymbol{w}}_3\widetilde{\mathbf{N}}^k + \widetilde{\boldsymbol{w}}_\alpha\Theta^k,_\alpha),_3\widetilde{\mathbf{N}}^k \\
&\qquad - \mathcal{G}^{\alpha\beta}\Theta^i,_\alpha(\widetilde{\boldsymbol{w}}_3\widetilde{\mathbf{N}}^k + \widetilde{\boldsymbol{w}}_\gamma\Theta^k,_\gamma),_\beta\widetilde{\mathbf{N}}^k\big]\Big|_{y_3=0} \\
&= \widetilde{\boldsymbol{w}}_\alpha\widetilde{\mathbf{N}}^i,_\alpha + \widetilde{\boldsymbol{w}}_{\alpha,3}\theta^i,_\alpha - g^{\alpha\beta}\theta^i,_\alpha(\widetilde{\boldsymbol{w}}_{3,\beta} - \widetilde{\boldsymbol{w}}_\gamma b_{\gamma\beta})\,.
\end{aligned} \tag{76}$$

Combining (75) and (76), we conclude the desired identity. □

With Lemma 4.1, we can now prove Theorem 1.2 with $\mathrm{n} = 3$; note that the case $\mathrm{n} = 2$ follows from the more general case by considering vectors of the type $\boldsymbol{u} = (\boldsymbol{u}^1(x_1,x_2), \boldsymbol{u}^2(x_1,x_2), 0)$.

Proof of Theorem 1.2. Let $u \in H^{\mathrm{k}+1}(\Omega)$, and $\operatorname{curl}\boldsymbol{u} = f$, $\operatorname{div}\boldsymbol{u} = g$, $\nabla_{\partial\Omega}\boldsymbol{u}\cdot\mathbf{N} = h$. By the well-known identity

$$-\Delta\boldsymbol{u} = \operatorname{curl}\operatorname{curl}\boldsymbol{u} - \nabla\operatorname{div}\boldsymbol{u} \qquad \text{in} \quad \Omega\,, \tag{77}$$

we find that if χ is a smooth cut-off function with $\operatorname{spt}(\chi) \subset\subset \Omega$, then $\chi\boldsymbol{u}$ satisfies

$$\begin{aligned} -\Delta(\chi\boldsymbol{u}) &= -\boldsymbol{u}\Delta\chi - 2\nabla\chi\cdot\nabla\boldsymbol{u} + \chi(\operatorname{curl}\boldsymbol{f} - \nabla g) \qquad && \text{in} \quad \mathrm{O}\,,\\ \chi\boldsymbol{u} &= 0 && \text{on} \quad \partial\mathrm{O}\,, \end{aligned}$$

for some smooth domain $\mathrm{O} \subset\subset \Omega$ (choose O to be some smooth domain so that $\operatorname{spt}(\chi) \subset\subset \mathrm{O} \subset\subset \Omega$). Standard interior elliptic estimates then show that

$$\|\chi\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\mathrm{O})} \leqslant C\Big[\|\boldsymbol{u}\|_{H^{\mathrm{k}}(\mathrm{O})} + \|\boldsymbol{f}\|_{H^{\mathrm{k}}(\Omega)} + \|g\|_{H^{\mathrm{k}}(\Omega)}\Big]\,. \tag{78}$$

Now we proceed to the estimates near the boundary. Let $\{\zeta_m\}_{m=1}^K$ and $\{\theta_m\}_{m=1}^K$ be a partition of unity (subordinate to $\mathcal{U}_m$) and charts satisfying

(1) $\theta_m : B(0,r_m) \to \mathcal{U}_m$ belongs to $H^{\mathrm{k}+1}(B(0,r_m))$;
(2) $\theta_m : B_+(0,r_m) \to \Omega\cap\mathcal{U}_m$;
(3) $\theta_m : B(0,r_m)\cap\{y_3 = 0\} \to \partial\Omega\cap\mathcal{U}_m$,

and g_m and b_m denote the induced metric tensor and second fundamental form, respectively. Then

$$\begin{aligned} -\Delta(\zeta_m\boldsymbol{u}) &= \zeta_m(\operatorname{curl}\boldsymbol{f} - \nabla g) - \boldsymbol{u}\Delta\zeta_m - 2\nabla\zeta_m\cdot\nabla\boldsymbol{u} \qquad && \text{in} \quad \mathcal{U}_m\,,\\ (\zeta_m\nabla_{\partial\Omega}\boldsymbol{u})\cdot\mathbf{N} &= \zeta_m h && \text{on} \quad \mathcal{U}_m\,. \end{aligned}$$

We define $\widetilde{\boldsymbol{u}}_m = u\circ\theta_m$, $\widetilde{\zeta}_m = \zeta_m\circ\theta_m$, $\widetilde{\mathbf{N}} = \mathbf{N}\circ\theta_m$, and $A = (\nabla\theta_m)^{-1}$, $J = \det(\nabla\theta_m)$, $\mathrm{g}_m = \det(g_m)$. Taking the composition of the equations above with map θ_m, by the Piola identity (27), we find that

$$-\big[JA^j_\ell A^k_\ell(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m),_k\big],_j = J\Big[\zeta_m(\operatorname{curl}\boldsymbol{f} - \nabla g) - \boldsymbol{u}\Delta\zeta_m - 2\nabla\zeta_m\cdot\nabla\boldsymbol{u}\Big]\circ\theta_m \qquad \text{in} \quad \mathrm{U}\,, \tag{79a}$$

$$(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_\sigma\widetilde{\mathbf{N}}^i = \widetilde{\zeta}_m,_\sigma\widetilde{\boldsymbol{u}}^i_m\widetilde{\mathbf{N}}^i + \xi_m(h\circ\theta_m) \qquad \text{on} \quad \partial\mathrm{U}\,, \tag{79b}$$

for some smooth domain U satisfying that $\operatorname{spt}(\widetilde{\zeta}_m) \subseteq \overline{\mathrm{U}}$ and $\operatorname{spt}(\xi_m)\cap\partial\mathrm{U} \subseteq \{y_3 = 0\}$.

The function $(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m),_\sigma$, where $\sigma = 1,\cdots,\mathrm{n}-1$, will be the fundamental (dependent) variable that we are going to estimate; however, in order to apply Theorem 3.5 we need to transform the boundary condition (79b) to a homogeneous one. This is done by introducing the function $\boldsymbol{\phi}_\sigma$ which is the solution to the elliptic equation

$$\begin{aligned} \boldsymbol{\phi}_\sigma - (JA^j_\ell A^k_\ell\boldsymbol{\phi}_\sigma,_k),_j &= \mathbf{0} && \text{in} \quad \mathrm{U}\,,\\ \boldsymbol{\phi}_\sigma,_k A^k_\ell JA^j_\ell\boldsymbol{n}_j &= \widetilde{\zeta}_m,_\sigma\widetilde{\boldsymbol{u}}^i_m JA^j_\ell\boldsymbol{n}_j + \sqrt{\mathrm{g}_m}\,\widetilde{\zeta}_m(h\circ\theta_m) \qquad && \text{on} \quad \partial\mathrm{U}\,, \end{aligned}$$

in which $\boldsymbol{n}$ is the outward-pointing unit normal to $\partial\mathrm{U}$, and then defining $\boldsymbol{w}^i_\sigma = (\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_\sigma - A^r_i\boldsymbol{\phi}_\sigma,_r$ as the new dependent variable of interest. Since $\sqrt{\mathrm{g}_m}\,\widetilde{\mathbf{N}} = JA^{\mathrm{T}}\boldsymbol{n}$ on $B(0,r_m)\cap\{y_3 = 0\}$,

$$\boldsymbol{w}_\sigma\cdot\widetilde{\mathbf{N}} = (\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}_m),_\sigma\cdot\widetilde{\mathbf{N}} - \frac{\boldsymbol{\phi}_\sigma,_k A^k_\ell JA^j_\ell\boldsymbol{n}_j}{\sqrt{\mathrm{g}_m}} = 0 \qquad \text{on} \quad \partial\mathrm{U}\,;$$

thus $\boldsymbol{w}_\sigma$ satisfies a homogeneous boundary condition.

Differentiating (79a) with respect to y_σ, with a^{jk} denoting $JA^j_\ell A^k_\ell$ we find that $\boldsymbol{w}_\sigma$ satisfies

$$\boldsymbol{w}_\sigma - \frac{\partial}{\partial y_j}\Big(a^{jk}\frac{\partial \boldsymbol{w}_\sigma}{\partial y_k}\Big) = \boldsymbol{F}_\sigma \qquad \text{in} \quad \mathrm{U}\,, \tag{80a}$$

$$\boldsymbol{w}_\sigma \cdot \widetilde{\mathbf{N}} = 0 \qquad \text{on} \quad \partial\mathrm{U}\,, \tag{80b}$$

where $\boldsymbol{F}_\sigma$ is given by

$$\begin{aligned}\boldsymbol{F}^i_\sigma &= \Big[J\big(\zeta_m(\operatorname{curl}\boldsymbol{f} - \nabla g) - \boldsymbol{u}\Delta\zeta_m - 2\nabla\zeta_m\cdot\nabla\boldsymbol{u}\big)\circ\theta_m\Big]^i,_\sigma \\ &\quad + \boldsymbol{w}^i_\sigma + \big[(JA^j_\ell A^k_\ell),_\sigma(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_k\big],_j + \big[JA^j_\ell A^k_\ell(A^r_i\boldsymbol{\phi}_\sigma,_r),_k\big],_j\end{aligned}$$

Moreover, by Lemma 4.1,

$$\begin{aligned}\Big[\mathrm{P}_{\mathbf{N}^\perp}\Big(\frac{\partial(\boldsymbol{w}_\sigma\circ\theta_m^{-1})}{\partial\mathbf{N}}\Big)\Big]\circ\theta_m &= \big[\operatorname{curl}(\boldsymbol{w}_\sigma\circ\theta_m^{-1})\times\mathbf{N}\big]\circ\theta_m + g_m^{\alpha\beta}\theta_m,_\sigma(\boldsymbol{w}_\sigma\cdot\widetilde{\mathbf{N}}),_\beta \\ &\quad - g_m^{\alpha\beta}g_m^{\gamma\delta}(\boldsymbol{w}_\sigma\cdot\theta,_\delta)b_{\gamma\beta}\,\theta,_\sigma\ ;\end{aligned}$$

thus using (80b) in the second term of the right-hand side, we obtain that

$$\mathrm{P}_{\widetilde{\mathbf{N}}^\perp}\Big(a^{jk}\frac{\partial\boldsymbol{w}_\sigma}{\partial x_k}\boldsymbol{n}_j\Big) = \sqrt{\mathrm{g}_m}\,\big[\operatorname{curl}(\boldsymbol{w}_\sigma\circ\theta_m^{-1})\times\mathbf{N}\big]\circ\theta_m - \sqrt{\mathrm{g}_m}\,g_m^{\alpha\beta}g_m^{\gamma\delta}(\boldsymbol{w}_\sigma\cdot\theta,_\delta)b_{\gamma\beta}\,\theta,_\sigma \quad \text{on} \quad \partial\mathrm{U}\,. \tag{81}$$

Since

$$\begin{aligned}\big[\operatorname{curl}(\boldsymbol{w}_\sigma\circ\theta_m^{-1})\times\mathbf{N}\big]^i\circ\theta_m &= \varepsilon_{ijk}\varepsilon_{jrs}A^\ell_r w^s_\sigma,_\ell\,\widetilde{\mathbf{N}}^k \\ &= A^\ell_k\big[(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_\sigma - A^r_i\boldsymbol{\phi}_\sigma,_r\big],_\ell\,\widetilde{\mathbf{N}}^k - A^\ell_i\big[(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^k_m),_\sigma - A^r_k\boldsymbol{\phi}_\sigma,_r\big],_\ell\,\widetilde{\mathbf{N}}^k \\ &= A^\ell_k(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_{\alpha\ell}\,\widetilde{\mathbf{N}}^k - A^\ell_i(\xi_m\widetilde{\boldsymbol{u}}^k_m),_{\alpha\ell}\,\widetilde{\mathbf{N}}^k - A^\ell_k(A^r_i\boldsymbol{\phi}_\sigma,_r),_\ell\,\widetilde{\mathbf{N}}^k + A^\ell_i(A^r_k\boldsymbol{\phi}_\sigma,_r),_\ell\,\widetilde{\mathbf{N}}^k\end{aligned}$$

and

$$\begin{aligned}\big[\widetilde{\zeta}_m\big(\operatorname{curl}\boldsymbol{u}\times\mathbf{N}\big)\circ\theta_m\big],_\sigma &= \varepsilon_{ijk}\varepsilon_{jrs}\big(\widetilde{\zeta}_m A^\ell_r\widetilde{\boldsymbol{u}}^s_m,_\ell\,\widetilde{\mathbf{N}}^k\big),_\sigma \\ &= \big[A^\ell_k(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_\ell\,\widetilde{\mathbf{N}}^k - A^\ell_i(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^k_m),_\ell\,\widetilde{\mathbf{N}}^k\big],_\sigma - \big[A^\ell_k\xi_m,_\ell\,\widetilde{\boldsymbol{u}}^i_m\widetilde{\mathbf{N}}^k - A^\ell_i\xi_m,_\ell\,\widetilde{\boldsymbol{u}}^k_m\widetilde{\mathbf{N}}^k\big],_\sigma \\ &= A^\ell_k(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_{\alpha\ell}\,\widetilde{\mathbf{N}}^k - A^\ell_i(\xi_m\widetilde{\boldsymbol{u}}^k_m),_{\alpha\ell}\,\widetilde{\mathbf{N}}^k - (A^\ell_k\widetilde{\mathbf{N}}^k),_\sigma(\xi_m\widetilde{\boldsymbol{u}}^i_m),_\ell \\ &\quad - (A^\ell_i\widetilde{\mathbf{N}}^k),_\sigma(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_\ell - \big[A^\ell_k\xi_m,_\ell\,\widetilde{\zeta}_m,_\ell\,\widetilde{\boldsymbol{u}}^i_m\widetilde{\mathbf{N}}^k - A^\ell_i\widetilde{\zeta}_m,_\ell\,\widetilde{\boldsymbol{u}}^k_m\widetilde{\mathbf{N}}^k\big],_\sigma\ ,\end{aligned}$$

we find that

$$\begin{aligned}&\big[\operatorname{curl}(\boldsymbol{w}_\sigma\circ\theta_m^{-1})\times\mathbf{N}\big]^i\circ\theta_m - \big[\widetilde{\zeta}_m\big(\operatorname{curl}\boldsymbol{u}\times\mathbf{N}\big)\circ\theta_m\big],_\sigma \\ &\quad = -A^\ell_k(A^r_i\boldsymbol{\phi}_\sigma,_r),_\ell\,\widetilde{\mathbf{N}}^k + A^\ell_i(A^r_k\boldsymbol{\phi}_\sigma,_r),_\ell\,\widetilde{\mathbf{N}}^k + (A^\ell_k\widetilde{\mathbf{N}}^k),_\sigma(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_\ell \\ &\qquad + (A^\ell_i\widetilde{\mathbf{N}}^k),_\sigma(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_\ell + (A^\ell_k\xi_m,_\ell\,\widetilde{\zeta}_m,_\ell\,\widetilde{\boldsymbol{u}}^i_m\widetilde{\mathbf{N}}^k),_\sigma + (A^\ell_i\widetilde{\zeta}_m,_\ell\,\widetilde{\boldsymbol{u}}^k_m\widetilde{\mathbf{N}}^k),_\sigma\ ;\end{aligned}$$

thus (81) implies that

$$\mathrm{P}_{\widetilde{\mathbf{N}}^\perp}\Big(a^{jk}\frac{\partial\boldsymbol{w}_\sigma}{\partial x_k}\boldsymbol{n}_j\Big) = \sqrt{\mathrm{g}_m}\,\boldsymbol{G}_\sigma \qquad \text{on} \quad \partial\mathrm{U}\,, \tag{80c}$$

where $\boldsymbol{G}_\sigma$ is given by

$$\begin{aligned}\boldsymbol{G}_\sigma &= \big[\widetilde{\zeta}_m(f\times\mathbf{N})\circ\theta_m\big],_\sigma - A^\ell_k(A^r_i\boldsymbol{\phi}_\sigma,_r),_\ell\,\widetilde{\mathbf{N}}^k + A^\ell_i(A^r_k\boldsymbol{\phi}_\sigma,_r),_\ell\,\widetilde{\mathbf{N}}^k \\ &\quad + (A^\ell_k\widetilde{\mathbf{N}}^k),_\sigma(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_\ell + (A^\ell_i\widetilde{\mathbf{N}}^k),_\sigma(\widetilde{\zeta}_m\widetilde{\boldsymbol{u}}^i_m),_\ell + (A^\ell_k\widetilde{\zeta}_m,_\ell\,\widetilde{\boldsymbol{u}}^i_m\widetilde{\mathbf{N}}^k),_\sigma \\ &\quad + (A^\ell_i\widetilde{\zeta}_m,_\ell\,\widetilde{\boldsymbol{u}}^k_m\widetilde{\mathbf{N}}^k),_\sigma - g_m^{\alpha\beta}g_m^{\gamma\delta}\big[(\xi_m\widetilde{\boldsymbol{u}}^i_m),_\sigma\,\theta^i_m,_\delta - A^r_i\boldsymbol{\phi}_\sigma,_r\,\theta^i_m,_\delta\big]b_{\gamma\beta}\,\theta,_\sigma\ .\end{aligned}$$

As a consequence, $\boldsymbol{w}_\sigma$ is the solution to equation (80), and Theorem 3.5 (with $\ell = \mathrm{k}-1$ and $\mathrm{w} = n$) then implies that $(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m),_\sigma$ satisfies

$$\|\boldsymbol{w}_\sigma\|_{H^{\mathrm{k}}(\mathrm{U})} \leqslant C\Big[\|\boldsymbol{F}_\sigma\|_{H^{\mathrm{k}-2}(\mathrm{U})} + \|\boldsymbol{G}_\sigma\|_{H^{\mathrm{k}-1.5}(\partial\mathrm{U})}\Big] \tag{82}$$

for some constant $C = C\big(\|a\|_{H^{\mathrm{k}}(B(0,r_m))}, \|A\|_{H^{\mathrm{k}}(B(0,r_m))}, \|\widetilde{\mathbf{N}}\|_{H^{\mathrm{k}-0.5}(\partial\mathrm{U})}\big)$.

We focus on the estimate of $\boldsymbol{F}_\sigma$ first. By Corollary 2.6,

$$\|J\|_{H^{\mathrm{k}}(B(0,r_m))} + \|A\|_{H^{\mathrm{k}}(B(0,r_m))} + \|\mathrm{g}_m\|_{H^{\mathrm{k}-0.5}(\partial\mathrm{U})} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\,; \tag{83}$$

thus Corollary 3.7 suggests that

$$\begin{aligned}\|\boldsymbol{\phi}_\sigma\|_{H^{\mathrm{k}+1}(\mathrm{U})} &\leqslant C(\|a\|_{H^{\mathrm{k}}(\mathrm{U})}, |\partial\Omega|_{H^{k+0.5}})\big\|\widetilde{\zeta}_m,_\sigma\,\widetilde{\boldsymbol{u}}^j_m J A^\ell_j \boldsymbol{n}_\ell + \sqrt{\mathrm{g}_m}\,\widetilde{\zeta}_m(h\circ\theta_m)\big\|_{H^{\mathrm{k}-0.5}(\partial\mathrm{U})} \\ &\leqslant C(|\partial\Omega|_{H^{k+0.5}})\Big[\|\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|h\|_{H^{k-0.5}(\partial\Omega)}\Big]. \end{aligned}\tag{84}$$

Moreover, by Corollary 2.7, we also have that

$$\|a\|_{H^{\mathrm{k}}(\mathrm{U})} + \|\widetilde{\mathbf{N}}\|_{H^{\mathrm{k}-0.5}(\partial\mathrm{U})} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\,. \tag{85}$$

As a consequence,

$$\|\boldsymbol{F}_\sigma\|_{H^{\mathrm{k}-2}(\mathrm{U})} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|\boldsymbol{f}\|_{H^{\mathrm{k}}(\Omega)} + \|g\|_{H^{\mathrm{k}}(\Omega)} + \|h\|_{H^{\mathrm{k}-0.5}(\partial\Omega)} + \|\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)}\Big]. \tag{86}$$

As for the estimate of $\boldsymbol{G}_\sigma$, the highest order terms are $(A^\ell_k \widetilde{\mathbf{N}}^k),_\sigma\,(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^i_m),_\ell$, $(A^\ell_i \widetilde{\mathbf{N}}^k),_\sigma\,(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^i_m),_\ell$ and $g^{\alpha\beta}_m g^{\gamma\delta}_m (\xi_m \widetilde{\boldsymbol{u}}^i_m),_\sigma\,\theta^i_m,_\delta\, b_{\gamma\beta}\,\theta,_\sigma$, and we apply (14) to obtain, for example, that

$$\begin{aligned}\big\|(A^\ell_k \widetilde{\mathbf{N}}^k),_\sigma\,(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^i_m),_\ell\,\|_{H^{\mathrm{k}-1.5}(\partial\mathrm{U})} &\leqslant C\big\|(A^\ell_k \widetilde{\mathbf{N}}^k),_\sigma\,(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}^i_m),_\ell\,\big\|_{H^{\mathrm{k}-1}(\mathrm{U})} \\ &\leqslant C\|\partial(A\widetilde{\mathbf{N}})\|_{H^{\mathrm{k}-1}(\mathrm{U})}\|\nabla(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m)\|_{H^s(\mathrm{U})} \leqslant C(|\partial|\Omega|_{2.5})\|\boldsymbol{u}\|_{H^{s+1}(\Omega)}\,,\end{aligned}$$

where $s = \max\big\{\mathrm{k}-1, \dfrac{\mathrm{k}}{2} + \dfrac{\mathrm{n}}{4}\big\}$ is chosen so that (14) can be applied (since $s > \dfrac{\mathrm{n}}{2}$). Therefore,

$$\|\boldsymbol{G}_\sigma\|_{H^{\mathrm{k}-1.5}(\partial\mathrm{U})} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|\boldsymbol{f}\|_{H^{\mathrm{k}-1}(\Omega)} + \|h\|_{H^{\mathrm{k}-0.5}(\partial\Omega)} + \|\boldsymbol{u}\|_{H^{s+1}(\Omega)}\Big]. \tag{87}$$

Combining estimates (82), (83), (84), (85), (86) and (87), we find that

$$\begin{aligned}\|(\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m),_\sigma\,\|_{H^{\mathrm{k}}(\mathrm{U})} &\leqslant \|\boldsymbol{w}_\sigma\|_{H^{\mathrm{k}}(\mathrm{U})} + \|A^{\mathrm{T}}\nabla\boldsymbol{\phi}_\sigma\|_{H^{\mathrm{k}}(\mathrm{U})} \\ &\leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|\boldsymbol{f}\|_{H^{\mathrm{k}}(\Omega)} + \|g\|_{H^{\mathrm{k}}(\Omega)} + \|h\|_{H^{\mathrm{k}-0.5}(\partial\Omega)} + \|\boldsymbol{u}\|_{H^{s+1}(\Omega)}\Big]. \end{aligned}\tag{88}$$

Finally, following the same procedure of Step 4 in the proof of Theorem 3.4 (that is, using (79a) to obtain an expression of $\widetilde{\zeta}_m\partial^{\mathrm{k}+1-j}\nabla^j\widetilde{\boldsymbol{u}}_m,_{33}$) and then arguing by induction on j, we find that

$$\|\widetilde{\zeta}_m \widetilde{\boldsymbol{u}}_m\|_{H^{\mathrm{k}+1}(\mathrm{U})} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|\boldsymbol{f}\|_{H^{\mathrm{k}}(\Omega)} + \|g\|_{H^{\mathrm{k}}(\Omega)} + \|h\|_{H^{\mathrm{k}-0.5}(\partial\Omega)} + \|\boldsymbol{u}\|_{H^{s+1}(\Omega)}\Big].$$

The estimate above and estimate (78) provide us with

$$\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\mathrm{U})} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|\boldsymbol{f}\|_{H^{\mathrm{k}}(\Omega)} + \|g\|_{H^{\mathrm{k}}(\Omega)} + \|h\|_{H^{\mathrm{k}-0.5}(\partial\Omega)} + \|\boldsymbol{u}\|_{H^{s+1}(\Omega)}\Big].$$

Since $0 < s+1 < \mathrm{k}+1$, by interpolation and Young's inequality,

$$\|\boldsymbol{u}\|_{H^s(\Omega)} \leqslant C_\delta\|\boldsymbol{u}\|_{L^2(\Omega)} + \delta\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \qquad \forall\,\delta > 0\,,$$

so by choosing $\delta > 0$ small enough we conclude (5). □

By studying the vector-valued elliptic equation (28), with the help of Theorem 3.10 we can also conclude (6).

Remark 4.2. *Suppose that Ω is a bounded $H^{\mathrm{k}+2}$-domain for some $\mathrm{k} > \frac{\mathrm{n}}{2}$. Since $\nabla_{\partial\Omega} \boldsymbol{u} \cdot \mathbf{N} = \nabla_{\partial\Omega}(\boldsymbol{u} \cdot \mathbf{N}) - \boldsymbol{u} \cdot \nabla_{\partial\Omega}\mathbf{N}$, by interpolation we find that*

$$\begin{aligned} \|\nabla_{\partial\Omega}\boldsymbol{u}\cdot\mathbf{N}\|_{H^{\mathrm{k}-0.5}(\partial\Omega)} &\leqslant \|\boldsymbol{u}\cdot\mathbf{N}\|_{H^{\mathrm{k}+0.5}(\partial\Omega)} + \|\boldsymbol{u}\cdot\nabla_{\partial\Omega}\mathbf{N}\|_{H^{\mathrm{k}-0.5}(\partial\Omega)} \\ &\leqslant \|\boldsymbol{u}\cdot\mathbf{N}\|_{H^{\mathrm{k}+0.5}(\partial\Omega)} + C(|\partial\Omega|_{H^{\mathrm{k}+1.5}})\|\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} \\ &\leqslant \|\boldsymbol{u}\cdot\mathbf{N}\|_{H^{\mathrm{k}+0.5}(\partial\Omega)} + C(|\partial\Omega|_{H^{\mathrm{k}+1.5}},\delta)\|\boldsymbol{u}\|_{L^2(\Omega)} + \delta\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)}. \end{aligned}$$

Hence, by choosing $\delta > 0$ small enough we conclude that there exists a generic constant $C = C(|\partial\Omega|_{H^{\mathrm{k}+1.5}})$ such that

$$\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\mathrm{curl}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\mathrm{div}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\boldsymbol{u}\cdot\mathbf{N}\|_{H^{\mathrm{k}+0.5}(\partial\Omega)}\Big].$$

Similarly, we also have that

$$\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \leqslant C\Big[\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\mathrm{curl}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\mathrm{div}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\boldsymbol{u}\times\mathbf{N}\|_{H^{\mathrm{k}+0.5}(\partial\Omega)}\Big]$$

for some constant $C = C(|\partial\Omega|_{H^{\mathrm{k}+1.5}})$.

5. The Proof of Theorem 1.1

We begin with the following problem: find $\boldsymbol{v}$ such that

$$\begin{aligned} \mathrm{curl}\boldsymbol{v} &= \boldsymbol{f} && \text{in} \quad \Omega\,, && (89a)\\ \mathrm{div}\boldsymbol{v} &= g && \text{in} \quad \Omega\,, && (89b)\\ \boldsymbol{v}\cdot\mathbf{N} &= h && \text{on} \quad \partial\Omega\,. && (89c) \end{aligned}$$

From the divergence theorem and the fact that $\mathrm{div}\,\mathrm{curl} = 0$, we assume that

$$\mathrm{div}\boldsymbol{f} = 0 \quad \text{and} \quad \int_\Omega g\,dx = \int_{\partial\Omega} h\,dS\,. \tag{90}$$

Since g and h satisfies the solvability condition (90), there exists a solution ϕ to the Poisson equation with Neumann boundary conditions:

$$\begin{aligned} \Delta\phi &= g && \text{in} \quad \Omega\,, && (91a)\\ \frac{\partial\phi}{\partial\mathbf{N}} &= h && \text{on} \quad \partial\Omega\,. && (91b) \end{aligned}$$

Let $\boldsymbol{u} = \boldsymbol{v} - \nabla\phi$. Then $\boldsymbol{u}$ satisfies

$$\begin{aligned} \mathrm{curl}\boldsymbol{u} &= \boldsymbol{f} && \text{in} \quad \Omega\,, && (92a)\\ \mathrm{div}\boldsymbol{u} &= 0 && \text{in} \quad \Omega\,, && (92b)\\ \boldsymbol{u}\cdot\mathbf{N} &= 0 && \text{on} \quad \partial\Omega\,. && (92c) \end{aligned}$$

Hence, if (92) is solvable, then there exists a solution to (89).

5.1. **Uniqueness of the solution.** We show that under reasonable conditions, the solution to (89) is unique. We first assume that Ω is a bounded convex domain. If $\boldsymbol{\varphi} \in \mathscr{C}^2(\Omega) \cap \mathscr{C}^1(\overline{\Omega})$, then for all

$\boldsymbol{u} \in H^1(\Omega)$,

$$\begin{aligned}\int_\Omega \operatorname{curl}\boldsymbol{u} \cdot \operatorname{curl}\boldsymbol{\varphi}\, dx &= \int_\Omega \boldsymbol{u} \cdot \operatorname{curl}\operatorname{curl}\boldsymbol{\varphi}\, dx + \int_{\partial\Omega} (\mathbf{N} \times \boldsymbol{u}) \cdot \operatorname{curl}\boldsymbol{\varphi}\, dS \\ &= \int_\Omega \boldsymbol{u} \cdot (-\Delta\boldsymbol{\varphi} + \nabla\operatorname{div}\boldsymbol{\varphi})\, dx + \int_{\partial\Omega} (\mathbf{N} \times \boldsymbol{u}) \cdot \operatorname{curl}\boldsymbol{\varphi}\, dS \\ &= \int_\Omega \boldsymbol{u} \cdot (-\Delta\boldsymbol{\varphi} + \nabla\operatorname{div}\boldsymbol{\varphi})\, dx + \int_{\partial\Omega} \Big[\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{N}} \cdot \boldsymbol{u} - \boldsymbol{u}^k \mathbf{N}_j \boldsymbol{\varphi}^j{,_k}\Big] dS \\ &= \int_\Omega (\nabla\boldsymbol{u} : \nabla\boldsymbol{\varphi} - \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{\varphi})\, dx + \int_{\partial\Omega} \Big[(\boldsymbol{u}\cdot\mathbf{N})\operatorname{div}\boldsymbol{\varphi} - \boldsymbol{u}^k \mathbf{N}_j \boldsymbol{\varphi}^j{,_k}\Big] dS\,.\end{aligned}$$

Using the notation introduced in the proof of Lemma 4.1, in any local chart $(\mathcal{U}, \theta)$ we have on $\partial\Omega$,

$$\begin{aligned}(\boldsymbol{u}^k \mathbf{N}_j \boldsymbol{\varphi}^j{,_k}) \circ \theta &= \tilde{\boldsymbol{u}}^k \tilde{\mathbf{N}}^j \big(\tilde{\boldsymbol{\varphi}}^j{,_\mathrm{n}} \tilde{\mathbf{N}}^k + g^{\alpha\beta} \tilde{\boldsymbol{\varphi}}^j{,_\alpha} \theta^k{,_\beta}\big) \\ &= (\tilde{\boldsymbol{u}} \cdot \tilde{\mathbf{N}})(\tilde{\boldsymbol{\varphi}}^j{,_\mathrm{n}} \tilde{\mathbf{N}}^j) + g^{\alpha\beta}(\tilde{\boldsymbol{u}} \cdot \theta{,_\beta})(\tilde{\boldsymbol{\varphi}} \cdot \tilde{\mathbf{N}}){,_\alpha} - g^{\alpha\beta}(\tilde{\boldsymbol{u}} \cdot \theta{,_\beta})\tilde{\mathbf{N}}{,_\alpha} \cdot \tilde{\boldsymbol{\varphi}} \\ &= (\tilde{\boldsymbol{u}} \cdot \tilde{\mathbf{N}})(\tilde{\boldsymbol{\varphi}}^j{,_\mathrm{n}} \tilde{\mathbf{N}}^j) + g^{\alpha\beta}(\tilde{\boldsymbol{u}} \cdot \theta{,_\beta})(\tilde{\boldsymbol{\varphi}} \cdot \tilde{\mathbf{N}}){,_\alpha} - g^{\alpha\beta} g^{\gamma\delta} b_{\alpha\gamma} (\tilde{\boldsymbol{u}} \cdot \theta{,_\beta})(\tilde{\boldsymbol{\varphi}} \cdot \theta{,_\delta})\,,\end{aligned}$$

so that using (96),

$$\begin{aligned}\int_\Omega \operatorname{curl}\boldsymbol{u} \cdot \operatorname{curl}\boldsymbol{\varphi}\, dx &= \int_\Omega (\nabla\boldsymbol{u} : \nabla\boldsymbol{\varphi} - \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{\varphi})\, dx \\ &\quad + \int_{\partial\Omega} (\boldsymbol{u}\cdot\mathbf{N})\Big[\operatorname{div}_{\partial\Omega}\boldsymbol{\varphi} + 2\mathrm{H}(\boldsymbol{\varphi}\cdot\mathbf{N})\Big] dS \\ &\quad + \sum_{m=1}^K \int_{\partial\Omega\cap\mathcal{U}_m} \zeta_m \big[g_m^{\alpha\beta} g_m^{\gamma\delta} b_{m\alpha\gamma} \big((\boldsymbol{u}\circ\theta_m)\cdot\theta{,_\beta}\big)\big((\boldsymbol{\varphi}\circ\theta_m)\cdot\theta{,_\delta}\big)\big] \circ \theta_m^{-1} dS \\ &\quad - \sum_{m=1}^K \int_{\partial\Omega\cap\mathcal{U}_m} \zeta_m \big[g_m^{\alpha\beta} \big((\boldsymbol{u}\circ\theta_m)\cdot\theta{,_\beta}\big)\big((\boldsymbol{\varphi}\cdot\mathbf{N})\circ\theta_m\big){,_\alpha}\big] \circ \theta_m^{-1} dS\,.\end{aligned} \tag{93}$$

Therefore, if $\boldsymbol{v}_1, \boldsymbol{v}_2 \in H^1(\Omega)$ are two solutions to (89), then $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ satisfies

$$\begin{aligned}&\|\operatorname{curl}\boldsymbol{v}\|^2_{L^2(\Omega)} \\ &\quad = \|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)} + \sum_{m=1}^K \int_{\partial\Omega\cap\mathcal{U}_m} \zeta_m \big[g_m^{\alpha\beta} g_m^{\gamma\delta} b_{m\alpha\gamma} \big((\boldsymbol{u}\circ\theta_m)\cdot\theta{,_\beta}\big)\big((\boldsymbol{\varphi}\circ\theta_m)\cdot\theta{,_\delta}\big)\big] \circ \theta_m^{-1} dS\,.\end{aligned}$$

Since Ω is convex, $g_m^{\alpha\beta} g_m^{\gamma\delta} b_{m\alpha\gamma}$ is non-negative definite for all m; thus the Poincaré inequality (23) shows that for some constant $c > 0$,

$$c\|\boldsymbol{v}\|^2_{H^1(\Omega)} \leqslant \|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)} \leqslant \|\operatorname{curl}\boldsymbol{v}\|^2_{L^2(\Omega)} = 0$$

which implies that $\boldsymbol{v} = 0$. In other words, the H^1-solution to (89) must be unique if Ω is bounded and convex.

Now suppose that Ω is a general bounded domain, and there are two solutions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in $H^{1+\epsilon}(\Omega)$ for some $\epsilon > 0$. Then $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ satisfies $\operatorname{curl}\boldsymbol{v} = 0$ in Ω. Since $\boldsymbol{v} \in H^{1+\epsilon}(\Omega)$, $\boldsymbol{v}$ has a trace on any one-dimensional (smooth) curve. By the Stokes theorem, for all $\Sigma \subseteq \Omega$ with piecewise $\mathscr{C}^1$-boundary $\partial\Sigma$,

$$\int_{\partial\Sigma} \boldsymbol{v} \cdot d\mathbf{r} = \int_\Sigma \operatorname{curl}\boldsymbol{v} \cdot \mathbf{N}\, dS = 0\,,$$

so $\phi(x) = \displaystyle\int_{C_p} \boldsymbol{v} \cdot d\mathbf{r}$, where C_p is a smooth curve connecting x and some fixed point p in Ω, is a well-defined scalar function. Moreover, $\nabla\phi = \boldsymbol{v}$. In other words, if Ω is connected, an $H^{1+\epsilon}$-solution

$\boldsymbol{v}$ to $\operatorname{curl}\boldsymbol{v} = 0$ must be the gradient of a scalar potential for some potential function ϕ. Since $\boldsymbol{v}$ also satisfies $\operatorname{div}\boldsymbol{v} = 0$ in Ω and $\boldsymbol{v} \cdot \mathbf{N} = 0$ on $\partial\Omega$, ϕ must be a constant which implies that $\boldsymbol{v} = 0$. As a consequence, the $H^{1+\epsilon}$-solution to (89) must be unique (in each connected component of Ω).

5.2. **Existence of solutions.** We solve (92) by finding a solution $\boldsymbol{u}$ of the form $\boldsymbol{u} = \operatorname{curl}\boldsymbol{w}$ for a divergence-free vector field $\boldsymbol{w}$. Indeed, ff such $\boldsymbol{w}$ exists, then using (77), $\boldsymbol{w}$ satisfies

$$-\Delta \boldsymbol{w} = \boldsymbol{f} \qquad \text{in} \quad \Omega\,, \tag{94a}$$

$$\operatorname{div}\boldsymbol{w} = 0 \qquad \text{in} \quad \Omega\,, \tag{94b}$$

$$\operatorname{curl}\boldsymbol{w} \cdot \mathbf{N} = 0 \qquad \text{on} \quad \partial\Omega\,. \tag{94c}$$

We note that if $\boldsymbol{w}$ is smooth, the divergence-free condition (94b) can indeed be treated as a boundary condition

$$\operatorname{div}\boldsymbol{w} = 0 \qquad \text{on} \quad \partial\Omega\,. \tag{94b'}$$

In fact, taking the divergence of (94a) we find that

$$\Delta\operatorname{div}\boldsymbol{w} = \operatorname{div}\boldsymbol{f} = 0 \qquad \text{in} \quad \Omega\,,$$

where we use the solvability condition (90) to establish the last equality; thus if $\boldsymbol{w}$ satisfies (94a,b'), $\boldsymbol{w}$ automatically has zero divergence. In other words, we may instead assume that $\boldsymbol{w}$ satisfies (94a,b',c). Our goal next is to find some suitable boundary condition to replace (94b',c).

Suppose that Ω is a bounded $\mathscr{C}^2$-domain of $\mathbb{R}^3$ (whose $\mathscr{C}^2$-regularity will eventually be relaxed). Following the notation introduced in the proof of Lemma 4.1, we find that

$$\begin{aligned}
\operatorname{div}\boldsymbol{w}\big|_{\partial\Omega} \circ \theta &= \big[\widetilde{\mathbf{N}}^k(\widetilde{\boldsymbol{w}}_3\widetilde{\mathbf{N}}^k + \widetilde{\boldsymbol{w}}_\alpha \Theta^k,_\alpha),_3 + \mathcal{G}^{\alpha\beta}\Theta^k,_\alpha(\widetilde{\boldsymbol{w}}_3\widetilde{\mathbf{N}}^k + \widetilde{\boldsymbol{w}}_\gamma\Theta^k,_\gamma),_\beta\big]\Big|_{y_3=0} \\
&= \widetilde{\boldsymbol{w}}_{3,3} + g^{\alpha\beta}\theta^k,_\alpha(\widetilde{\boldsymbol{w}}_{\gamma,\beta}\,\theta^k,_\gamma + \widetilde{\boldsymbol{w}}_\gamma\theta^k,_{\beta\gamma} + \widetilde{\mathbf{N}}^k,_\beta\,\widetilde{\boldsymbol{w}}_3 + \widetilde{\mathbf{N}}^k\widetilde{\boldsymbol{w}}_{3,\beta}) \\
&= \widetilde{\boldsymbol{w}}_{3,3} + \widetilde{\boldsymbol{w}}_{\gamma,\gamma} + \Gamma^\beta_{\beta\gamma}\widetilde{\boldsymbol{w}}_\gamma + 2\mathrm{H}\widetilde{\boldsymbol{w}}_3\,,
\end{aligned}$$

where $\Gamma^\gamma_{\alpha\beta}$ is the Christoffel symbol defined by

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}\,g^{\delta\delta}(g_{\alpha\delta,\beta} + g_{\beta\delta,\alpha} - g_{\alpha\beta,\delta}) = g^{\gamma\delta}\theta,_{\alpha\beta}\cdot\theta,_\delta\ ,$$

and $\mathrm{H} = \dfrac{1}{2}\,g^{\alpha\beta}b_{\alpha\beta}$ is the mean curvature of $\partial\Omega$. Let $\operatorname{div}_{\partial\Omega}$ denote the divergence operator on $\partial\Omega$ given by

$$(\operatorname{div}_{\partial\Omega}\boldsymbol{v})\circ\theta = \widetilde{\boldsymbol{v}}_{\gamma,\gamma} + \Gamma^\beta_{\beta\gamma}\widetilde{\boldsymbol{v}}_\gamma \qquad \forall\ \boldsymbol{v}\in \mathrm{T}(\partial\Omega)\ (\text{or equivalently, } \widetilde{\boldsymbol{v}}^i = \widetilde{\boldsymbol{v}}_\gamma\theta^i,_\gamma)\,.$$

Then

$$\operatorname{div}\boldsymbol{w} = \big[\widetilde{\boldsymbol{w}}_{3,3} + \operatorname{div}_{\partial\Omega}(\mathrm{P}_{\widetilde{\mathbf{N}}^\perp}\widetilde{\boldsymbol{w}}) + 2\widetilde{\mathrm{H}}\widetilde{\boldsymbol{w}}_3\big]\circ\theta^{-1} \qquad \text{on} \quad \partial\Omega\,,$$

where we recall that $\mathrm{P}_{\mathbf{N}^\perp}$ denotes the projection of a vector onto the tangent plane of $\partial\Omega$. With the help of (74), for all local charts $(\mathcal{U},\theta)$,

$$\Big[\frac{\partial\boldsymbol{w}}{\partial\mathbf{N}}\cdot\mathbf{N}\Big]\circ\theta = \big[\widetilde{\boldsymbol{w}}_{3,3}\,\widetilde{\mathbf{N}}^i + \widetilde{\boldsymbol{w}}_{\alpha,3}\theta^i,_\alpha + \widetilde{\boldsymbol{w}}_\alpha g^{\alpha\beta}\widetilde{\mathbf{N}}^i,_\beta\big]\widetilde{\mathbf{N}}^i = \widetilde{\boldsymbol{w}}_{3,3}\ ; \tag{95}$$

thus

$$\operatorname{div}\boldsymbol{w} = \frac{\partial\boldsymbol{w}}{\partial\mathbf{N}}\cdot\mathbf{N} + 2\mathrm{H}(\boldsymbol{w}\cdot\mathbf{N}) + \operatorname{div}_{\partial\Omega}(\mathrm{P}_{\mathbf{N}^\perp}\boldsymbol{w}) \qquad \text{on} \quad \partial\Omega\,. \tag{96}$$

On the other hand, if $\mathbf{N} = \dfrac{1}{\sqrt{\mathrm{g}}}(\theta,_1 \times \theta,_2)$, using the permutation symbol we find that

$$\begin{aligned}(\mathrm{curl}\boldsymbol{w}\cdot\mathbf{N})\circ\theta &= \varepsilon_{ijk}(w^k,_j\,\mathbf{N}^i)\circ\theta = \varepsilon_{ijk}\big[\widetilde{\mathbf{N}}^j\widetilde{\boldsymbol{w}}^k_{,3} + g^{\alpha\beta}\theta^j,_\alpha\,\widetilde{\boldsymbol{w}}^k,_\beta\big]\widetilde{\mathbf{N}}^i\\ &= \varepsilon_{ijk}\theta^j,_\alpha\,\widetilde{\mathbf{N}}^i g^{\alpha\beta}\widetilde{\boldsymbol{w}}^k,_\beta = (\widetilde{\mathbf{N}}\times\theta,_\alpha)\cdot g^{\alpha\beta}\widetilde{\boldsymbol{w}},_\beta\\ &= \sqrt{\mathrm{g}}(\delta-\alpha)g^{\alpha\beta}g^{\gamma\delta}\theta^k,_\gamma\,\widetilde{\boldsymbol{w}}^k,_\beta\ ,\end{aligned}$$

where we have used that $\mathbf{N}\times\theta,_\alpha = \sqrt{\mathrm{g}}(\delta-\alpha)g^{\gamma\delta}\theta,_\gamma$ to establish the last equality. Writing the sum explicitly, we find that

$$\begin{aligned}(\mathrm{curl}\boldsymbol{w}\cdot\mathbf{N})\circ\theta &= \sqrt{\mathrm{g}}(g^{11}g^{22} - g^{12}g^{12})(\theta,_2\cdot\widetilde{\boldsymbol{w}},_1 - \theta,_1\cdot\widetilde{\boldsymbol{w}},_2)\\ &= \frac{1}{\sqrt{\mathrm{g}}}(\theta,_2\cdot\widetilde{\boldsymbol{w}},_1 - \theta,_1\cdot\widetilde{\boldsymbol{w}},_2)\,.\end{aligned}$$

Since $\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{w}}_3\widetilde{\mathbf{N}} + \mathrm{P}_{\widetilde{\mathbf{N}}^\perp}\widetilde{\boldsymbol{w}}$,

$$\begin{aligned}\sqrt{\mathrm{g}}(\mathrm{curl}\boldsymbol{w}\cdot\mathbf{N})\circ\theta &= \theta,_2\cdot(\mathrm{P}_{\widetilde{\mathbf{N}}^\perp}\widetilde{\boldsymbol{w}} + \widetilde{\boldsymbol{w}}_3\mathbf{N}),_1 - \theta,_1\cdot(\mathrm{P}_{\widetilde{\mathbf{N}}^\perp}\widetilde{\boldsymbol{w}} + \widetilde{\boldsymbol{w}}_3\widetilde{\mathbf{N}}),_2\\ &= \widetilde{\boldsymbol{w}}_{\alpha,1}g_{2\alpha} + \widetilde{\boldsymbol{w}}_\alpha\Gamma^\beta_{1\alpha}g_{\beta2} - \widetilde{\boldsymbol{w}}_{\alpha,2}g_{1\alpha} - \widetilde{\boldsymbol{w}}_\alpha\Gamma^\beta_{2\alpha}g_{\beta1}\\ &= \theta,_2\cdot\nabla_{\theta,_1}(\mathrm{P}_{\widetilde{\mathbf{N}}^\perp}\widetilde{\boldsymbol{w}}) - \theta,_1\cdot\nabla_{\theta,_2}(\mathrm{P}_{\widetilde{\mathbf{N}}^\perp}\widetilde{\boldsymbol{w}})\,,\end{aligned} \tag{97}$$

where for two tangent vector field $X = X^\alpha\theta,_\alpha$ and $Y = Y^\beta\theta,_\beta$ on a two-dimensional manifold, $\nabla_X Y$ is the covariant derivative of Y in the direction X given by

$$\nabla_X Y = X^\beta(Y^\alpha,_\beta + Y^\gamma\Gamma^\alpha_{\gamma\beta})\,\theta,_\alpha\ .$$

5.2.1. *The case that* $\Omega = B(0,R)$. Now we assume that $\Omega = B(0,R)$ for some $R > 0$. Having obtained (96) and (97), in order to achieve (94b',c) it is natural to consider the case $\mathrm{P}_{\mathbf{N}^\perp}\boldsymbol{w} = 0$. In other words, we consider the following elliptic problem (with a non-standard boundary condition)

$$-\Delta\boldsymbol{w} = \boldsymbol{f} \qquad \text{in} \quad \Omega\,, \tag{98a}$$

$$\mathrm{P}_{\mathbf{N}^\perp}\boldsymbol{w} = \mathbf{0} \qquad \text{on} \quad \partial\Omega\,, \tag{98b}$$

$$\frac{\partial\boldsymbol{w}}{\partial\mathbf{N}}\cdot\mathbf{N} + 2\,\mathrm{H}(\boldsymbol{w}\cdot\mathbf{N}) = 0 \qquad \text{on} \quad \partial\Omega\,, \tag{98c}$$

where we remark that $\mathrm{H} = R^{-1}$ is a positive constant. We also note that (94b') and (94c) are direct consequence of (98b,c), and (95) suggests that (98c) is in fact a Robin boundary condition for $\widetilde{\boldsymbol{w}}_3$. The goal is to find a solution to (98) in the Hilbert space

$$H^1_\tau(\Omega) \equiv \big\{\boldsymbol{w}\in H^1(\Omega)\,\big|\,\mathrm{P}_{\mathbf{N}^\perp}\boldsymbol{w} = 0\big\} = \big\{\boldsymbol{w}\in H^1(\Omega)\,\big|\,\boldsymbol{w}\times\mathbf{N} = 0\big\}\,.$$

In order to solve (98), we find the weak formulation first, and this amounts to computing $\displaystyle\int_{\partial\Omega}\frac{\partial\boldsymbol{w}}{\partial\mathbf{N}}\cdot\boldsymbol{\varphi}\,dS$. Nevertheless, if $\boldsymbol{\varphi}\in H^1_\tau(\Omega)$, then $\boldsymbol{\varphi} = (\boldsymbol{\varphi}\cdot\mathbf{N})\mathbf{N}$; thus, if $\boldsymbol{w}$ satisfies (98c), for all $\boldsymbol{\varphi}\in H^1_\tau(\Omega)$ we have

$$-\int_{\partial\Omega}\frac{\partial\boldsymbol{w}}{\partial\mathbf{N}}\cdot\boldsymbol{\varphi}\,dS = -\int_{\partial\Omega}\Big[\frac{\partial\boldsymbol{w}}{\partial\mathbf{N}}\cdot\mathbf{N}\Big](\boldsymbol{\varphi}\cdot\mathbf{N})\,dS = 2\int_{\partial\Omega}\mathrm{H}(\boldsymbol{w}\cdot\mathbf{N})(\boldsymbol{\varphi}\cdot\mathbf{N})\,dS\,.$$

The identity above implies the following

Definition 5.1. *A vector-valued function* $\boldsymbol{w}\in H^1_\tau(\Omega)$ *is said to be a weak solution to* (98) *if*

$$\int_\Omega\nabla\boldsymbol{w}:\nabla\boldsymbol{\varphi}\,dx + 2\int_{\partial\Omega}\mathrm{H}(\boldsymbol{w}\cdot\mathbf{N})(\boldsymbol{\varphi}\cdot\mathbf{N})\,dS = (\boldsymbol{f},\boldsymbol{\varphi})_{L^2(\Omega)} \quad \forall\,\boldsymbol{\varphi}\in H^1_\tau(\Omega)\,, \tag{99}$$

where $\nabla\boldsymbol{w}:\nabla\boldsymbol{\varphi} = \boldsymbol{w}^i,_j\,\boldsymbol{\varphi}^i,_j$.

Since $\mathrm{H} > 0$, the left-hand side of (99) obviously defines a bounded, coercive bilinear form on $H^1_\tau(\Omega) \times H^1_\tau(\Omega)$. In fact, using Poincaré's inequality (23) we find that for some generic constant $c > 0$,

$$\int_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{w} dx + 2\int_{\partial\Omega} \mathrm{H}(\boldsymbol{w}\cdot\mathbf{N})(\boldsymbol{w}\cdot\mathbf{N})\, dS \geqslant c\|\boldsymbol{w}\|^2_{H^1(\Omega)} \qquad \boldsymbol{w} \in H^1_\tau(\Omega)\,;$$

hence by the Lax-Milgram theorem, there exists a unique $\boldsymbol{w} \in H^1_\tau(\Omega)$ satisfying the weak formulation (99) and the basic energy estimate

$$\|\boldsymbol{w}\|_{H^1(\Omega)} \leqslant C\|\boldsymbol{f}\|_{L^2(\Omega)}\,. \tag{100}$$

Before proceeding, we establish the corresponding regularity theory for equation (98).

Lemma 5.2. *Let $\Omega = B(0,R) \subseteq \mathbb{R}^3$ for some $R > 0$. Then for all $\boldsymbol{f} \in H^{\ell-1}(\Omega)$ for some $\ell \geqslant 1$, the weak solution $\boldsymbol{w}$ to (98) in fact belongs to $H^{\ell+1}(\Omega)$, and satisfies*

$$\|\boldsymbol{w}\|_{H^{\ell+1}(\Omega)} \leqslant C\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)}\,. \tag{101}$$

Proof. As the proof of Theorem 3.4 we prove this lemma by induction. The weak solution $\boldsymbol{w}$ indeed belongs to $H^1(\Omega)$ satisfies (100). Assume that $\boldsymbol{w} \in H^j(\Omega)$ for some $j \leqslant \ell$. If χ is a smooth cut-off function so that $\mathrm{spt}(\chi) \subset\subset \Omega$, the same computation as in the proof of Theorem 3.4 (with $a^{jk} = \delta^{jk}$) suggests that

$$\|\chi D^{j+1}\boldsymbol{w}\|_{L^2(\Omega)} \leqslant C\Big[\|\boldsymbol{f}\|_{H^{j-1}(\Omega)} + \|\boldsymbol{w}\|_{H^j(\Omega)}\Big]\,, \tag{102}$$

where the constant C depends on the distance between the support of χ and $\partial\Omega$.

Now we focus on the estimate of $\boldsymbol{w}$ near $\partial\Omega$. Let $\{\zeta_m, \mathcal{U}_m, \theta_m\}_{m=1}^K$ be defined as in the proof of Theorem 1.2, and $g_{\alpha\beta} = \theta_{m,\alpha}\cdot\theta_{m,\beta}$. Define

$$\boldsymbol{\varphi}_1 = (-1)^j \zeta_m \mathbf{N}\big[\Lambda_\epsilon \partial^{2j}\Lambda_\epsilon(\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big]\circ\theta_m^{-1}\,,$$

where $\widetilde{\zeta}_m = \zeta_m\circ\theta_m$, $\widetilde{\boldsymbol{w}}_m = \boldsymbol{w}\circ\theta$ and $\widetilde{\mathbf{N}} = \mathbf{N}\circ\theta$. Since $\mathrm{P}_{\mathbf{N}^\perp}\boldsymbol{\varphi}_1 = 0$, $\boldsymbol{\varphi}_1$ can be used as a test function in (99). Similar to the computations in Step 2 in the proof of Theorem 3.4, we find that

$$\begin{aligned}\int_\Omega \nabla\boldsymbol{w} : \nabla\boldsymbol{\varphi}_1\, dx &\geqslant \frac{1}{2}\big\|\nabla\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|^2_{L^2(B_m^+)} - C\|\boldsymbol{u}\|_{H^j(\Omega)}\Big[\big\|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|_{H^1(B_m^+)} + \|\boldsymbol{u}\|_{H^j(\Omega)}\Big]\\ &\geqslant \frac{1}{2}\big\|\nabla\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|^2_{L^2(B_m^+)} - C_\delta\|\boldsymbol{w}\|^2_{H^j(\Omega)} - \delta\big\|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|_{H^1(B_m^+)}\,, \end{aligned}\tag{103}$$

Now we focus on the term $\displaystyle\int_{\partial\Omega}\mathrm{H}(\boldsymbol{w}\cdot\mathbf{N})(\boldsymbol{\varphi}_1\cdot\mathbf{N})\,dS$. Making a change of variable and integrating by parts, we find that

$$\begin{aligned}\int_{\partial\Omega}\mathrm{H}(\boldsymbol{w}\cdot\mathbf{N})(\boldsymbol{\varphi}_1\cdot\mathbf{N})\,dS &= \int_{\{y_3=0\}}\partial^j\Lambda_\epsilon\big[\sqrt{\mathrm{g}}\,\mathrm{H}(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big]\partial^j\big(\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big)\,dS\\ &= \int_{\{y_3=0\}}\partial\Lambda_\epsilon\big[\sqrt{\mathrm{g}}\,\mathrm{H}\partial^{j-1}(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big]\partial^j\big(\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big)\,dS\\ &\quad+\sum_{k=1}^{j-1}\binom{j-1}{k}\int_{\{y_3=0\}}\partial\Lambda_\epsilon\big[\partial^k(\sqrt{\mathrm{g}}\,\mathrm{H})\partial^{j-1-k}(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big]\partial^j\big(\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big)\,dS\,.\end{aligned}$$

Using our commutator notation,

$$\begin{aligned}\int_{\partial\Omega} \mathrm{H}(\boldsymbol{w}\cdot\mathbf{N})(\boldsymbol{\varphi}_1\cdot\mathbf{N})\,dS &= \int_{\{y_3=0\}} \sqrt{\mathrm{g}}\,\mathrm{H}\big|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big|^2 dS \\ &+ \int_{\{y_3=0\}} \big[\partial(\sqrt{\mathrm{g}}\,\mathrm{H})\partial^{j-1}\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big]\partial^j\big(\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big)\,dS \\ &+ \int_{\{y_3=0\}} \partial\big[[\![\Lambda_\epsilon,\sqrt{\mathrm{g}}\,\mathrm{H}]\!]\partial^{j-1}(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big]\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\,dS \\ &+ \sum_{k=1}^{j-1}\binom{j-1}{k}\int_{\{y_3=0\}} \Lambda_\epsilon\big[\partial^k(\sqrt{\mathrm{g}}\,\mathrm{H})\partial^{j-k}(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big]\partial^j\big(\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big)\,dS \\ &+ \sum_{k=1}^{j-1}\binom{j-1}{k}\int_{\{y_3=0\}} \Lambda_\epsilon\big[\partial^{k+1}(\sqrt{\mathrm{g}}\,\mathrm{H})\partial^{j-1-k}(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big]\partial^j\big(\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big)\,dS\,.\end{aligned} \tag{104}$$

The commutator estimate (26) and interpolation, as well as Young's inequality, suggest that

$$\begin{aligned}&\int_{\{y_3=0\}} \partial\big[[\![\Lambda_\epsilon,\sqrt{\mathrm{g}}\,\mathrm{H}]\!]\partial^{j-1}(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big]\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\,dS \\ &\quad\geqslant -C\big\|\partial^{j-1}\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|_{L^2(\{y_3=0\})}\big\|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|_{L^2(\{y_3=0\})} \\ &\quad\geqslant -C_\delta\|\boldsymbol{w}\|^2_{H^j(\Omega)} - \delta\big\|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|^2_{H^1(B(0,r_m))}\,.\end{aligned}$$

Using Hölder's inequality to estimate the other terms we obtain that

$$\begin{aligned}\int_{\partial\Omega} \mathrm{H}(\boldsymbol{w}\cdot\mathbf{N})(\boldsymbol{\varphi}_1\cdot\mathbf{N})\,dS &\geqslant \int_{\{y_3=0\}} \sqrt{\mathrm{g}}\,\mathrm{H}\big|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big|^2 dS \\ &\quad - C_\delta\|\boldsymbol{w}\|^2_{H^j(\Omega)} - \delta\big\|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|^2_{H^1(B(0,r_m))}.\end{aligned} \tag{105}$$

Moreover, it is easy to see that

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}_1\,dx &\leqslant C\|\boldsymbol{f}\|_{H^j(\Omega)}\big\|\partial^{j+1}\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|_{L^2(B(0,r_m))} \\ &\leqslant C_\delta\|\boldsymbol{f}\|^2_{L^2(\Omega)} + \delta\big\|\partial^{j+1}\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|^2_{L^2(B(0,r_m))}\,.\end{aligned} \tag{106}$$

Combining (103), (105) and (106),

$$\begin{aligned}&\big\|\nabla\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|^2_{L^2(B_m^+)} + \int_{\{y_3=0\}} \sqrt{\mathrm{g}}\,\mathrm{H}\big|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big|^2 dS \\ &\qquad\leqslant C_\delta\Big[\|\boldsymbol{f}\|^2_{L^2(\Omega)} + \|\boldsymbol{w}\|^2_{H^j(\Omega)}\Big] + \delta\big\|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|^2_{H^1(B(0,r_m))}.\end{aligned} \tag{107}$$

Using Poincaré's inequality, there exists a constant $c>0$ such that

$$c\|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\|^2_{H^1(\Omega)} \leqslant \big\|\nabla\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|^2_{L^2(B_m^+)} + \int_{\{y_3=0\}} \sqrt{\mathrm{g}}\,\mathrm{H}\big|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big|^2 dS\,,$$

so by choosing $\delta>0$ small enough we find that

$$\big\|\partial^j\Lambda_\epsilon(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|_{H^1(B_m^+)} \leqslant C\Big[\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{w}\|_{H^j(\Omega)}\Big]\,.$$

Since the right-hand side of the estimate above is independent of ϵ, we can pass ϵ to the limit and obtain that

$$\big\|\partial^j(\widetilde{\zeta}_m\widetilde{\boldsymbol{w}}_m\cdot\widetilde{\mathbf{N}})\big\|_{H^1(B_m^+)} \leqslant C\Big[\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{w}\|_{H^j(\Omega)}\Big]\,. \tag{108}$$

The estimate above provides the regularity of $\boldsymbol{w}$ in the normal direction.

To see the regularity of the tangential component of $\boldsymbol{w}$, an alternative test function has to be employed. Define

$$\boldsymbol{\varphi}_2 = (-1)^j \zeta_m \mathbf{N} \times \big[\Lambda_\epsilon \partial^{2j} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})\big] \circ \theta_m^{-1} \,.$$

We note that since $\boldsymbol{w} \times \mathbf{N} = 0$ on $\partial\Omega$, $\boldsymbol{\varphi}_2 = 0$ on $\partial\Omega$ so $\boldsymbol{\varphi}_2$ may be used as a test function. Since $\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w}$, with J and A denoting $\det(\nabla\theta_m)$ and $(\nabla\theta_m)^{-1}$ respectively, using b^{rs} to denote $JA^r_k A^s_k$ we find that

$$\begin{aligned}
\int_\Omega \nabla\boldsymbol{w} : \nabla\boldsymbol{\varphi}_2\, dx &= (-1)^j \int_{B_m^+} b^{rs} \widetilde{\boldsymbol{w}}^i_m,_s \big[\widetilde{\zeta}_m \widetilde{\mathbf{N}} \times \Lambda_\epsilon \partial^{2j} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})\big]^i,_r\, dy \\
&= (-1)^j \int_{B_m^+} b^{rs} (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}^i_m,_s \times \widetilde{\mathbf{N}})^i \Lambda_\epsilon \partial^{2j} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_r\, dy \\
&\quad + (-1)^j \int_{B_m^+} b^{rs} \widetilde{\boldsymbol{w}}^i_m,_s \big[(\widetilde{\zeta}_m \widetilde{\mathbf{N}}),_r \times \Lambda_\epsilon \partial^{2j} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})\big]^i dy \\
&= \int_{B_m^+} \partial^j \Lambda_\epsilon \big[b^{rs} (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_s\big] \partial^j \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_r\, dy \\
&\quad - \int_{B_m^+} \partial^j \Lambda_\epsilon \big[b^{rs} \big((\widetilde{\zeta}_m,_s \widetilde{\boldsymbol{w}},_m \times \widetilde{\mathbf{N}})^i + (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}},_m \times \widetilde{\mathbf{N}},_s)^i\big)\big] \partial^j \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_r\, dy \\
&\quad + \int_{B_m^+} \partial^j \Lambda_\epsilon b^{rs} \big[\widetilde{\boldsymbol{w}}^i_m,_s \times (\widetilde{\zeta}_m \widetilde{\mathbf{N}}),_r\big] \partial^j \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i dy \,.
\end{aligned}$$

Similar to the procedure of deriving (42), by Leibniz's rule,

$$\begin{aligned}
&\int_{B_m^+} \partial^j \Lambda_\epsilon b^{rs} (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_s \,\partial^j \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_r\, dy \\
&\qquad = \int_{B_m^+} b^{rs} \partial^j \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_s \,\partial^j \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_r\, dy \\
&\qquad\quad + \int_{B_m^+} \partial b^{rs} \partial^{j-1} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_s \,\partial^j \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_r\, dy \\
&\qquad\quad + \int_{B_m^+} \partial \left[\!\left[b^{rs}, \Lambda_\epsilon \right]\!\right] \partial^{j-1} (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_s \,\partial^j \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_r\, dy \\
&\qquad\quad + \sum_{k=0}^{j-2} \binom{j-1}{k} \int_{B_m^+} \partial \Lambda_\epsilon \big[\partial^{j-1-k} b^{rs} \partial^k (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_s\big] \partial^j \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})^i,_r\, dy \,.
\end{aligned}$$

Since $\{\theta_m\}_{m=1}^M$ is chosen so that $A \approx \mathrm{Id}$, b^{rs} is positive-definitive. As a consequence, by the commutator estimate (26) and Young's inequality,

$$\begin{aligned}
\int_\Omega \nabla\boldsymbol{w} : \nabla\boldsymbol{\varphi}_2\, dx &\geqslant \frac{1}{2} \big\|\partial^j \nabla \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})\big\|^2_{L^2(B_m^+)} \\
&\quad - C \big\|\partial^{j-1} \nabla \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})\big\|_{L^2(B_m^+)} \big\|\partial^j \nabla \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})\big\|_{L^2(B_m^+)} \\
&\geqslant \frac{1}{4} \big\|\partial^j \nabla \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})\big\|^2_{L^2(B_m^+)} - C\|\boldsymbol{w}\|^2_{H^j(\Omega)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}_2\, dx &\leqslant C\|\boldsymbol{f}\|_{H^{j-1}(\Omega)} \big\|\partial^{j+1} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})\big\|_{L^2(B(0,r_m))} \\
&\leqslant C_\delta \|\boldsymbol{f}\|^2_{L^2(\Omega)} + \delta \big\|\partial^{j+1} \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}})\big\|^2_{L^2(B(0,r_m))} \,;
\end{aligned}$$

thus using φ_2 as a test function in (99) and choosing $\delta > 0$ small enough, by the fact that $\int_{\partial\Omega} \mathrm{H}(\boldsymbol{w} \cdot \mathbf{N})(\varphi_2 \cdot \mathbf{N})\, dS = 0$ we conclude that

$$\big\| \partial^j \Lambda_\epsilon (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}}) \big\|_{H^1(B_m^+)} \leqslant C \Big[\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{w}\|_{H^j(\Omega)} \Big] . \tag{109}$$

Since the right-hand side is ϵ-independent, we can pass ϵ to the limit and obtain that

$$\big\| \partial^j (\widetilde{\zeta}_m \widetilde{\boldsymbol{w}}_m \times \widetilde{\mathbf{N}}) \big\|_{H^1(B_m^+)} \leqslant C \Big[\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{w}\|_{H^j(\Omega)} \Big] . \tag{110}$$

The estimate above provides the regularity of $\boldsymbol{w}$ in the tangential direction.

Since every vector u can be expressed as $\boldsymbol{u} = \widetilde{\mathbf{N}} \times (\boldsymbol{u} \times \widetilde{\mathbf{N}}) + (\boldsymbol{u} \cdot \widetilde{\mathbf{N}})\widetilde{\mathbf{N}}$, the combination of (108) and (110) then shows that

$$\big\| \widetilde{\zeta}_m \partial^j \widetilde{\boldsymbol{w}}_m \big\|_{H^1(B_m^+)} \leqslant C \Big[\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{w}\|_{H^j(\Omega)} \Big] .$$

Finally, we follow Step 4 of Theorem 3.4 or Step 3 of Theorem 3.5 to conclude that

$$\big\| \widetilde{\zeta}_m \nabla^j \widetilde{\boldsymbol{w}}_m \big\|_{H^1(B_m^+)} \leqslant C \Big[\|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{w}\|_{H^j(\Omega)} \Big] . \tag{111}$$

Estimate (101) is concluded from combining the H^1-estimate (100), the interior estimate (102) and the boundary estimate (111). ▫

5.2.2. *The case that Ω is a general H^{k+1}-domain.* If Ω is a general H^{k+1}-domain, the mean curvature H can be negative on some portion of $\partial\Omega$, leading to a problematic Robin boundary condition (with the wrong sign) (98c). To overcome this difficulty, we instead consider a similar problem defined on a ball containing Ω.

Let $B(0, R)$ be an open ball so that $\Omega \subset\subset B(0, R)$, and $\boldsymbol{F}$ be a divergence-free vector field on $B(0, R)$ so that $\boldsymbol{F} = \boldsymbol{f}$ in $\overline{\Omega}$; that is, $\boldsymbol{F}$ is a divergence-free extension of $\boldsymbol{f}$. If $\boldsymbol{f} \in L^2(\Omega)$, such an $\boldsymbol{F}$ (in $B(0, R)\backslash\Omega$) can be obtained by first solving the elliptic equation

$$\Delta\phi = 0 \qquad \text{in} \quad B(0, R)\backslash\Omega\,, \tag{112a}$$

$$\frac{\partial\phi}{\partial\mathbf{N}} = \boldsymbol{f} \cdot \mathbf{N} \qquad \text{on} \quad \partial\Omega\,, \tag{112b}$$

$$\frac{\partial\phi}{\partial\mathbf{N}} = 0 \qquad \text{on} \quad \partial\Omega\,, \tag{112c}$$

and setting $\boldsymbol{F} = \nabla\phi$ on $B(0, R)\backslash\Omega$. We note that $\boldsymbol{F} \in L^2(B(0, R))$ even if $\boldsymbol{f} \in H^{\ell-1}(\Omega)$; thus $\boldsymbol{F}$ must be less regular than $\boldsymbol{f}$ due to the lack of continuity of the derivatives of $\boldsymbol{F}$ across $\partial\Omega$.

Now consider

$$-\Delta\boldsymbol{w} = \boldsymbol{F} \qquad \text{in} \quad B(0, R)\,, \tag{113a}$$

$$\mathrm{P}_{\mathbf{N}^\perp}\boldsymbol{w} = \mathbf{0} \qquad \text{on} \quad \partial B(0, R)\,, \tag{113b}$$

$$\frac{\partial\boldsymbol{w}}{\partial\mathbf{N}} \cdot \mathbf{N} + 2\,\mathrm{H}(\boldsymbol{w} \cdot \mathbf{N}) = 0 \qquad \text{on} \quad \partial B(0, R)\,. \tag{113c}$$

By Lemma 5.2, there exists a strong solution $\boldsymbol{w} \in H^2(B(0, R))$ to (113) (so that (113) also holds in the pointwise sense).

Now we show that $\boldsymbol{w}$ has zero divergence. Let $d = \mathrm{div}\boldsymbol{w} \in H^1(B(0, R))$. We claim that d is a weak solution to

$$\Delta d = 0 \qquad \text{in} \quad B(0, R)\,, \tag{114a}$$

$$d = 0 \qquad \text{on} \quad \partial B(0, R)\,; \tag{114b}$$

that is, $d \in H^1_0(B(0,R))$ and d satisfies

$$\int_{B(0,R)} \nabla d \cdot \nabla \varphi \, dx = 0 \qquad \forall\, \varphi \in H^1_0(B(0,R))\,. \tag{115}$$

The boundary condition $d = 0$ on $\partial B(0,R)$ is obvious because of (96) and (113b,c). To see (115), we note that it suffices to show $\Delta d = 0$ in the sense of distribution since $\mathcal{D}(B(0,R))$ is dense in $H^1_0(B(0,R))$. Let $\varphi \in \mathcal{D}(B(0,R))$, and define $\boldsymbol{\psi} = \nabla\varphi$. Then $\boldsymbol{\psi} \in \mathcal{D}(B(0,R))$, and

$$\begin{aligned} -\int_{B(0,R)} \Delta \boldsymbol{w} \cdot \boldsymbol{\psi} \, dx &= \int_{B(0,R)} \boldsymbol{F} \cdot \boldsymbol{\psi} \, dx = \int_{B(0,R)\setminus\Omega} \boldsymbol{F} \cdot \nabla\varphi \, dx + \int_{\Omega} \boldsymbol{f} \cdot \nabla\varphi \, dx \\ &= \int_{\partial\Omega} \Big(\boldsymbol{f} \cdot \mathbf{N} - \frac{\partial \phi}{\partial \mathbf{N}}\Big)\, \varphi \, dS = 0\,. \end{aligned}$$

On the other hand, since $\boldsymbol{w} \in H^2(B(0,R))$, we have $d \in H^1(B(0,R))$ and

$$\begin{aligned} -\int_{B(0,R)} \Delta \boldsymbol{w} \cdot \boldsymbol{\psi} \, dx &= \int_{B(0,R)} \nabla \boldsymbol{w} : \nabla \boldsymbol{\psi} \, dx = \int_{B(0,R)} \boldsymbol{w}^i{}_{,j}\, \varphi_{,ij} \, dx \\ &= -\int_{B(0,R)} d_{,j}\, \varphi_{,j} \, dx = -\int_{B(0,R)} \nabla d \cdot \nabla\varphi \, dx\,; \end{aligned}$$

thus we conclude (115). Therefore, d is the weak solution to (114) and so d must vanish in Ω which implies that $\operatorname{div}\boldsymbol{w} = 0$ in Ω. Finally, since $\boldsymbol{w} \in H^2(\Omega)$, applying (77) we find that $\boldsymbol{v} = \operatorname{curl}\boldsymbol{w} \in H^1(\Omega)$ satisfies $\operatorname{curl}\boldsymbol{v} = \boldsymbol{f}$ in Ω.

So far we have shown that there exists $\boldsymbol{v} \in H^1(B(0,R))$ satisfying

$$\begin{aligned} \operatorname{curl}\boldsymbol{v} &= \boldsymbol{F} && \text{in} \quad B(0,R)\,, \\ \operatorname{div}\boldsymbol{v} &= 0 && \text{in} \quad B(0,R)\,, \\ \boldsymbol{v} \cdot \mathbf{N} &= 0 && \text{on} \quad \partial B(0,R)\,, \end{aligned}$$

which in particular suggests that $\operatorname{curl}\boldsymbol{v} = \boldsymbol{f}$ in Ω. It is not clear that if $\boldsymbol{v}$ possesses better regularity since $\boldsymbol{v}$ is constructed using a non-smooth forcing $\boldsymbol{F}$. Let p be the $H^{\min\{3,\ell+1\}}$-solution to the elliptic equation

$$\begin{aligned} \Delta p &= 0 && \text{in} \quad \Omega\,, \\ \frac{\partial p}{\partial \mathbf{N}} &= -\boldsymbol{v} \cdot \mathbf{N} && \text{on} \quad \partial\Omega\,, \end{aligned}$$

and define $\boldsymbol{u} = \boldsymbol{v} + \nabla p$, then $\boldsymbol{u}$ is a solution to (92). We note that $\boldsymbol{u} \in H^1(\Omega)$ and satisfies

$$\begin{aligned} \|\boldsymbol{u}\|_{H^1(\Omega)} &\leqslant \|\boldsymbol{v}\|_{H^1(\Omega)} + \|\nabla p\|_{H^1(\Omega)} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\|\boldsymbol{w}\|_{H^2(\Omega)} \\ &\leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\|\boldsymbol{f}\|_{L^2(\Omega)}\,. \end{aligned} \tag{116}$$

In the following lemma, we show that the singularity of $\boldsymbol{v}$ in fact "cancels-out" the singularity of ∇p so that $\boldsymbol{u}$ possesses H^ℓ-regularity if $\boldsymbol{f} \in H^{\ell-1}(\Omega)$ for some $\ell \geqslant 2$.

Lemma 5.3. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded $H^{\mathrm{k}+1}$-domain for some $\mathrm{k} > \dfrac{3}{2}$. Then for all $\boldsymbol{f} \in H^{\ell-1}(\Omega)$ for some $1 \leqslant \ell \leqslant \mathrm{k}$, there exists a solution $\boldsymbol{u} \in H^\ell(\Omega)$ to (92) satisfying*

$$\|\boldsymbol{u}\|_{H^\ell(\Omega)} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)}\,. \tag{117}$$

Moreover, the solution is unique if Ω is convex or $\ell \geqslant 2$.

Proof. We again show that $u \in H^\ell(\Omega)$ by induction. We have shown the validity of the lemma for the case that $\ell = 1$. Now suppose that $\ell \geqslant 2$ and $u \in H^j(\Omega)$ for some $j \leqslant \ell - 1$. Since $\boldsymbol{u} = \text{curl}\boldsymbol{w} \in H^1(\Omega)$ satisfies (92b,c), using (93) we find that $\boldsymbol{u}$ satisfies

$$\int_\Omega \text{curl}\boldsymbol{u} \cdot \text{curl}\boldsymbol{\varphi}\, dx = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi}\, dx - \int_{\partial\Omega} \boldsymbol{h} \cdot \boldsymbol{\varphi}\, dS \qquad \forall\, \boldsymbol{\varphi} \in H^1_n(\Omega)\,,$$

where in local chart $(\mathcal{U}, \theta)$ $\boldsymbol{h}$ is given by $\boldsymbol{h} \circ \theta = -g^{\alpha\beta} g^{\gamma\delta} \big[(\boldsymbol{u} \circ \theta) \cdot \theta,_\beta\big] b_{\alpha\gamma}\, \theta,_\delta$. On the other hand,

$$\begin{aligned}\int_\Omega \boldsymbol{f} \cdot \text{curl}\boldsymbol{\varphi}\, dx &= \int_\Omega \varepsilon_{ijk} \boldsymbol{f}^i \boldsymbol{\varphi}^k,_j\, dx = -\int_\Omega \varepsilon_{ijk} \boldsymbol{f}^i,_j\, \boldsymbol{\varphi}^k dx + \int_{\partial\Omega} \varepsilon_{ijk} \boldsymbol{f}^i \mathbf{N}_j \boldsymbol{\varphi}^k dS \\ &= \int_\Omega \text{curl}\boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx + \int_{\partial\Omega} (\boldsymbol{f} \times \mathbf{N}) \cdot \boldsymbol{\varphi}\, dS \qquad \forall\, \boldsymbol{\varphi} \in H^1(\Omega).\end{aligned}$$

Using (92a), we find that u satisfies

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi}\, dx = \int_\Omega \text{curl}\boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx + \int_{\partial\Omega} (\boldsymbol{f} \times \mathbf{N} + \boldsymbol{h}) \cdot \boldsymbol{\varphi}\, dS \qquad \forall\, \boldsymbol{\varphi} \in H^1_n(\Omega)\,;$$

thus $\boldsymbol{u}$ is a weak solution to

$$-\Delta \boldsymbol{u} = \text{curl}\,\boldsymbol{f} \qquad \text{in} \quad \Omega\,, \tag{118a}$$
$$\boldsymbol{u} \cdot \mathbf{N} = 0 \qquad \text{on} \quad \partial\Omega\,, \tag{118b}$$
$$\mathrm{P}_{\mathbf{N}^\perp}\Big(\frac{\partial \boldsymbol{u}}{\partial \mathbf{N}} - \boldsymbol{f} \times \mathbf{N} - \boldsymbol{h}\Big) = \mathbf{0} \qquad \text{on} \quad \partial\Omega\,, \tag{118c}$$

Let us first assume that $\mathrm{k} \geqslant 3$. Then $\mathrm{k} - 1.5 > 1 = \dfrac{2}{2}$. Moreover, $j - 0.5 \leqslant \mathrm{k} - 1.5$; thus Proposition 2.4 suggests that

$$\|\boldsymbol{h}\|_{H^{j-0.5}(\partial\Omega)} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}}) \|b\|_{H^{\mathrm{k}-1.5}(\{y_3=0\})} \|\boldsymbol{u}\|_{H^{j-0.5}(\partial\Omega)} \leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}}) \|\boldsymbol{u}\|_{H^j(\Omega)}\,. \tag{119}$$

Therefore, by Corollary 3.6 (with $a^{jk} = \delta^{jk}$ and $\mathrm{w} = \mathbf{N}$) we conclude that

$$\begin{aligned}\|\boldsymbol{u}\|_{H^{j+1}(\Omega)} &\leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}}) \Big[\|\text{curl}\,\boldsymbol{f}\|_{H^{j-1}(\Omega)} + \|\boldsymbol{f} \times \mathbf{N} + \boldsymbol{h}\|_{H^{j-0.5}(\partial\Omega)}\Big] \\ &\leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}}) \Big[\|\boldsymbol{f}\|_{H^j(\Omega)} + \|\boldsymbol{u}\|_{H^j(\Omega)}\Big]\end{aligned}$$

which implies $\boldsymbol{u} \in H^{j+1}(\Omega)$. Estimate (117) then is concluded from estimate (116), interpolation and Young's inequality.

The case that $\mathrm{k} = 2$ (and $\ell = 2$) is a bit tricky. In this case (119) cannot be applied since $b, \boldsymbol{u}$ both belong to $H^{0.5}(\partial\Omega)$ while $H^{0.5}(\partial\Omega)$ is not a multiplicative algebra. To see why $\boldsymbol{u}$ indeed belongs to $H^2(\Omega)$ if $\boldsymbol{f} \in H^1(\Omega)$, let $\boldsymbol{u}^\epsilon$ to be the solution to

$$\lambda \boldsymbol{u}^\epsilon - \Delta \boldsymbol{u}^\epsilon = \text{curl}\,\boldsymbol{f} + \lambda \boldsymbol{u} \qquad \text{in} \quad \Omega\,, \tag{120a}$$
$$\boldsymbol{u}^\epsilon \cdot \mathbf{N} = 0 \qquad \text{on} \quad \partial\Omega\,, \tag{120b}$$
$$\mathrm{P}_{\mathbf{N}^\perp}\Big(\frac{\partial \boldsymbol{u}^\epsilon}{\partial \mathbf{N}} - \boldsymbol{f} \times \mathbf{N} - \boldsymbol{h}_\epsilon\Big) = \mathbf{0} \qquad \text{on} \quad \partial\Omega\,, \tag{120c}$$

where $\boldsymbol{u}$ on the right-hand side of (120a) is the solution to (92), $\boldsymbol{h}_\epsilon$ is a smooth version of $\boldsymbol{h}$ given by

$$\boldsymbol{h}_\epsilon = -\sum_{m=1}^K \zeta_m \Big[g_m^{\alpha\beta} g_m^{\gamma\delta} \big((\boldsymbol{u}^\epsilon \circ \theta_m) \cdot \theta_m,_\beta\big) (\Lambda_\epsilon b_{m\alpha\gamma})\, \theta_m,_\delta\Big] \circ \theta_m^{-1}$$

in which Λ_ϵ is the horizontal convolution defined in Section 2.4, and $\lambda \gg 1$ is a big constant so that the bilinear form

$$B(\boldsymbol{u}^\epsilon, \boldsymbol{\varphi}) = \lambda(\boldsymbol{u}^\epsilon, \boldsymbol{\varphi})_{L^2(\Omega)} + (\nabla \boldsymbol{u}^\epsilon, \nabla \boldsymbol{\varphi})_{L^2(\Omega)} + \int_{\partial\Omega} \boldsymbol{h}_\varepsilon \cdot \boldsymbol{\varphi}\, dS$$

is coercive on $H^1_n(\Omega) \times H^1_n(\Omega)$. Since $\Lambda_\epsilon b_m$ is smooth, we find that $\boldsymbol{h}_\epsilon \in H^{0.5}(\partial\Omega)$ satisfying

$$\begin{aligned}\|\boldsymbol{h}_\epsilon\|_{H^{0.5}(\partial\Omega)} &\leqslant C(|\partial\Omega|_{H^{2.5}})\Big[\|\partial_y\theta_m\|^6_{H^{1.25}(B(0,r_m)\cap\{y_3=0\})}\|\Lambda_\epsilon b_m\|_{H^{1.25}(B(0,r_m)\cap\{y_3=0\})}\|\boldsymbol{u}^\epsilon\|_{H^{0.5}(\partial\Omega)}\Big] \\ &\leqslant C_\epsilon\|\boldsymbol{u}^\epsilon\|_{H^1(\Omega)} \leqslant C_\epsilon\Big[\|\boldsymbol{f}\|_{L^2(\Omega)} + \lambda\|\boldsymbol{u}\|_{L^2(\Omega)}\Big] \leqslant C_\epsilon\|\boldsymbol{f}\|_{L^2(\Omega)}\,,\end{aligned}$$

where the dependence on ϵ in the constant C_ϵ is due to the horizontal convolution Λ_ϵ. As a sequence, $\boldsymbol{u}^\epsilon \in H^2(\Omega)$, and this fact further suggests that $\boldsymbol{h}_\epsilon$ satisfies

$$\begin{aligned}\|\boldsymbol{h}_\epsilon\|_{H^{0.5}(\partial\Omega)} &\leqslant C(|\partial\Omega|_{H^{2.5}})\Big[\|\partial_y\theta_m\|^6_{H^{1.25}(B(0,r_m)\cap\{y_3=0\})}\|b_m\|_{H^{0.5}(B(0,r_m)\cap\{y_3=0\})}\|\boldsymbol{u}^\epsilon\|_{H^{1.25}(\partial\Omega)}\Big] \\ &\leqslant C(|\partial\Omega|_{H^{2.5}})\|\boldsymbol{u}^\epsilon\|_{H^{1.75}(\Omega)} \leqslant C(|\partial\Omega|_{H^{2.5}})\|\boldsymbol{u}^\epsilon\|^{\frac14}_{H^1(\Omega)}\|\boldsymbol{u}^\epsilon\|^{\frac34}_{H^2(\Omega)}\,.\end{aligned}$$

By Young's inequality, we find that that $\boldsymbol{u}^\epsilon$ satisfies

$$\begin{aligned}\|\boldsymbol{u}^\epsilon\|_{H^2(\Omega)} &\leqslant C(|\partial\Omega|)\Big[\|\mathrm{curl}\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{f}\times\mathbf{N} + \boldsymbol{h}_\epsilon\|_{H^{0.5}(\partial\Omega)}\Big] \\ &\leqslant C(|\partial\Omega|_{H^{2.5}},\delta)\|\boldsymbol{f}\|_{H^1(\Omega)} + \delta\|\boldsymbol{u}^\epsilon\|_{H^2(\Omega)}\,.\end{aligned}$$

Choosing $\delta > 0$ small enough, we conclude that $\boldsymbol{u}^\epsilon$ has a uniform H^2 upper bound and possesses an H^1 convergent subsequence $\boldsymbol{u}^{\epsilon_j}$ with limit $\boldsymbol{v}$. This limit $\boldsymbol{v}$ must be $\boldsymbol{u}$ since $\boldsymbol{u}$ is also a weak solution to (120) and the strong solution to (120) is unique (by the Lax-Milgram theorem). Moreover, $\boldsymbol{u}$ satisfies (117) (for $\ell = 2$). □

Lemma 5.3 together with the elliptic estimate

$$\|\nabla\phi\|_{H^{j+1}(\Omega)} \leqslant C\big[\|g\|_{H^j(\Omega)} + \|h\|_{H^{j-0.5}(\partial\Omega)}\big]$$

for the solution ϕ to (91) then concludes the first part of Theorem 1.1.

5.3. **Solutions with prescribed tangential trace.** Having considered the boundary condition $\boldsymbol{v}\cdot\mathbf{N} = h$, we now establish the existence and uniqueness of the following problem:

$$\mathrm{curl}\boldsymbol{v} = \boldsymbol{f} \quad \text{in} \quad \Omega\,, \tag{121a}$$
$$\mathrm{div}\boldsymbol{v} = g \quad \text{in} \quad \Omega\,, \tag{121b}$$
$$\boldsymbol{v}\times\mathbf{N} = \boldsymbol{h} \quad \text{on} \quad \partial\Omega\,, \tag{121c}$$

in which (121c) prescribes the tangential trace of $\boldsymbol{v}$. We impose the following conditions on the forcing functions $\boldsymbol{f}$ and $\boldsymbol{h}$:

$$\mathrm{div}\boldsymbol{f} = 0 \ \text{ in } \ \Omega \quad \text{and} \quad \boldsymbol{h}\cdot\mathbf{N} = 0 \ \text{ on } \ \partial\Omega\,. \tag{122a}$$

For (121) to have a solution, one additional solvability condition has to be imposed. Let $\boldsymbol{u}$ be a solution to (92). Then $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ satisfies

$$\mathrm{curl}\boldsymbol{w} = \mathbf{0} \quad \text{in} \quad \Omega\,, \tag{123a}$$
$$\mathrm{div}\boldsymbol{w} = g \quad \text{in} \quad \Omega\,, \tag{123b}$$
$$\boldsymbol{w}\times\mathbf{N} = \boldsymbol{h} - \boldsymbol{u}\times\mathbf{N} \quad \text{on} \quad \partial\Omega\,. \tag{123c}$$

Taking the cross product of $\mathbf{N}$ with (123c), we find that

$$\boldsymbol{w} - (\boldsymbol{w}\cdot\mathbf{N})\mathbf{N} = \mathbf{N}\times\boldsymbol{h} - \big[\boldsymbol{u} - (\boldsymbol{u}\cdot\mathbf{N})\mathbf{N}\big] \qquad \text{on} \quad \partial\Omega\,.$$

If C is a closed curve on $\partial\Omega$ enclosing a surface $\Sigma \subseteq \partial\Omega$ so that $C = \partial\Sigma$ with a parameterization $\mathbf{r}$, then the Stokes theorem implies that

$$\begin{aligned} 0 &= \int_\Sigma \operatorname{curl}\boldsymbol{w}\cdot\mathbf{N}\,dS = \oint_C \boldsymbol{w}\cdot d\mathbf{r} = \oint_C [\boldsymbol{w} - (\boldsymbol{w}\cdot\mathbf{N})\mathbf{N}]\cdot d\mathbf{r} = \oint_C (\mathbf{N}\times\boldsymbol{h} - \boldsymbol{u})\cdot d\mathbf{r} \\ &= \oint_C (\mathbf{N}\times\boldsymbol{h})\cdot d\mathbf{r} - \int_\Sigma \operatorname{curl}\boldsymbol{u}\cdot\mathbf{N}\,dS = \oint_C (\mathbf{N}\times\boldsymbol{h})\cdot d\mathbf{r} - \int_\Sigma \boldsymbol{f}\cdot\mathbf{N}\,dS\,. \end{aligned}$$

Therefore, if Σ is a subset of $\partial\Omega$ with a piecewise smooth boundary,

$$\int_\Sigma \boldsymbol{f}\cdot\mathbf{N}\,dS = \oint_{\partial\Sigma} (\mathbf{N}\times\boldsymbol{h})\cdot d\mathbf{r}. \tag{122b}$$

(122a) and (122b) constitute the solvability conditions for equation (121).

5.3.1. *Uniqueness of solutions.* We first assume that Ω is convex. Using (93), we find that if $\boldsymbol{v}_1, \boldsymbol{v}_2 \in H^1(\Omega)$ are two solutions to (121), then $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ satisfies

$$0 = \|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)} + 2\int_{\partial\Omega} \mathrm{H}|\boldsymbol{v}\cdot\mathbf{N}|^2 dS\,;$$

thus if $\mathrm{H} \geqslant 0$, $\boldsymbol{v} = 0$ by the Poincaré inequality (23). Therefore, the H^1-solution to (121) must be unique if Ω is bounded and convex.

Now suppose that Ω is a general domain. Similar to Section 5.1, when looking for solutions in $H^{1+\epsilon}(\Omega)$ for some $\epsilon > 0$, the difference of two solutions must be the gradient of an $H^{2+\epsilon}$-scalar potential ϕ, and ϕ has to satisfy

$$\Delta\phi = 0 \qquad \text{in} \quad \Omega\,, \tag{124a}$$

$$\nabla\phi\times\mathbf{N} = \mathbf{0} \qquad \text{on} \quad \partial\Omega\,. \tag{124b}$$

Note that (124b) implies that the derivative of ϕ in all tangential direction on $\partial\Omega$ is zero. It follows that ϕ is constant on $\partial\Omega$. Consequently, ϕ is constant in Ω which implies the uniqueness of the solution to (121) in the space $H^{1+\epsilon}(\Omega)$.

5.3.2. *Existence of solutions.* In order to establish existence of solutions to (123), we look for a solution $\boldsymbol{w}$ of the form $\boldsymbol{w} = \nabla p$. We first assume that $\partial\Omega$ is path connected. By the Stokes theorem, the solvability condition (122b) shows that the function b given by

$$b(x) = \oint_{C_x} (\mathbf{N}\times\boldsymbol{h} - \boldsymbol{u})\cdot d\mathbf{r} \quad \forall\, x\in\partial\Omega\,, \tag{125}$$

where C_x is a smooth curve connecting a fixed point $x_0 \in \partial\Omega$ and $x \in \partial\Omega$, is independent of the curve C_x, and so is a well-defined function.

Since b is defined on $\partial\Omega$, b can be differentiated in all tangential directions. Moreover,

$$\nabla b\cdot\mathbf{T} = (\mathbf{N}\times\boldsymbol{h} - \boldsymbol{u})\cdot\mathbf{T} \qquad \text{on} \quad \partial\Omega$$

for all tangent vectors $\mathbf{T}$. Therefore, using (122a) we find that

$$\nabla b\times\mathbf{N} = (\mathbf{N}\times\boldsymbol{h} - \boldsymbol{u})\times\mathbf{N} = \boldsymbol{h} - \boldsymbol{u}\times\mathbf{N} \qquad \text{on} \quad \partial\Omega\,. \tag{126}$$

We remark that only the directional derivative of b in the tangential direction is necessary in order to compute $\nabla b\times\mathbf{N}$. Moreover, because of (126), $b \in H^{\ell-0.5}(\partial\Omega)$ and b satisfies

$$\begin{aligned} \|b\|_{H^{\ell-0.5}(\partial\Omega)} &\leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|\boldsymbol{h}\|_{H^{\ell-0.5}(\partial\Omega)} + \|\boldsymbol{u}\|_{H^{\ell-0.5}(\partial\Omega)}\Big] \\ &\leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{h}\|_{H^{\ell-0.5}(\partial\Omega)}\Big]\,. \end{aligned}$$

Let p be the solution of the elliptic equation

$$\Delta p = g \quad \text{in} \quad \Omega\,, \tag{127a}$$
$$p = b \quad \text{on} \quad \partial\Omega\,. \tag{127b}$$

We note that by Corollary 3.7, $p \in H^{\ell+1}(\Omega)$ satisfies the estimate

$$\begin{aligned}\|p\|_{H^{\ell+1}(\Omega)} &\leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|g\|_{H^{\ell-1}(\Omega)} + \|b\|_{H^{\ell-0.5}(\partial\Omega)}\Big]\\ &\leqslant C(|\partial\Omega|_{H^{\mathrm{k}+0.5}})\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega)} + \|g\|_{H^{\ell-1}(\Omega)} + \|\boldsymbol{h}\|_{H^{\ell-0.5}(\partial\Omega)}\Big]\,.\end{aligned}$$

Moreover, since $p = b$ on $\partial\Omega$, $\nabla(p-b)\times\mathbf{N} = 0$; hence,

$$\nabla p \times \mathbf{N} = \nabla b \times \mathbf{N} = \boldsymbol{h} - \boldsymbol{u}\times\mathbf{N} \qquad \text{on} \quad \partial\Omega\,. \tag{128}$$

As a consequence, (127a) and (128) show that $\boldsymbol{w} = \nabla p$ satisfies (123).

Now suppose that $\partial\Omega$ is not path connected. In this case we can define b on each connected component, and then solve (127) using such a b. We have, thus, proved the second part of Theorem 1.1.

6. The Proof of Theorem 1.5

Now we proceed to the proof of Theorem 1.5. We only prove (9) since the proof of (10) is similar. By assumption $\partial\Omega$ is in a small tubular neighborhood of the normal bundle over $\partial\mathcal{D}$; hence, there is height function $h(x,t)$ such that each point on $\partial\Omega$ is given by $x + h(x)\boldsymbol{n}(x)$, $x \in \partial\mathcal{D}$, where $\boldsymbol{n}$ is the outward-pointing unit normal to $\partial\mathcal{D}$. Let $\psi : \mathcal{D} \to \mathbb{R}^2$ solve

$$\begin{aligned}\Delta\psi &= 0 && \text{in} \quad \mathcal{D}\,,\\ \psi &= e + h\,\boldsymbol{n} && \text{on} \quad \partial\mathcal{D}\,,\end{aligned}$$

where e is the identity map. Then $\psi : \partial\mathcal{D} \to \partial\Omega$, and standard elliptic estimates show that for some constant $C = C(|\partial\mathcal{D}|_{H^{\mathrm{k}+0.5}})$,

$$\|\nabla\psi - \mathrm{Id}\|_{H^{\mathrm{k}}(\mathcal{D})} \leqslant C\|h\|_{H^{\mathrm{k}+0.5}(\partial\mathcal{D})} \leqslant C\epsilon \ll 1 \tag{129}$$

which further suggests that $\psi : \mathcal{D} \to \Omega$ is an $H^{\mathrm{k}+1}$-diffeomorphism since $\|d\|_{H^{\mathrm{k}+0.5}(\partial\mathcal{D})} < \epsilon \ll 1$. We note that according to the proofs of Corollary 2.6 and Corollary 2.8, there exists generic constants c_1 and C_1 independent of $|\partial\Omega|_{H^{\mathrm{k}+0.5}}$ such that if $j \leqslant k+1$,

$$c_1(1-\epsilon)\|f\|_{H^j(\Omega)} \leqslant \|f\circ\psi\|_{H^j(\mathcal{D})} \leqslant C_1(1+\epsilon)\|f\|_{H^j(\Omega)} \qquad \forall\, f \in H^j(\Omega)\,. \tag{130}$$

As a consequence, letting $A = (\nabla\psi)^{-1}$ we obtain that

$$\begin{aligned}\|(\mathrm{curl}\boldsymbol{u})\circ\psi\|_{H^{\mathrm{k}}(\mathcal{D})} &= \|\varepsilon_{ijk}A^r_j(\boldsymbol{u}^k\circ\psi),_r\|_{H^{\mathrm{k}}(\mathcal{D})} = \|\varepsilon_{ijk}(A^r_j - \delta^r_j)(\boldsymbol{u}^k\circ\psi),_r + \varepsilon_{ijk}(\boldsymbol{u}^k\circ\psi),_j\|_{H^{\mathrm{k}}(\mathcal{D})}\\ &\geqslant \|\mathrm{curl}(\boldsymbol{u}\circ\psi)\|_{H^{\mathrm{k}}(\mathcal{D})} - C\|A - \mathrm{Id}\|_{H^{\mathrm{k}}(\mathcal{D})}\|\boldsymbol{u}\circ\psi\|_{H^{\mathrm{k}+1}(\mathcal{D})}\,,\end{aligned}$$

where the constant $C = C(|\partial\mathcal{D}|_{H^{\mathrm{k}+0.5}})$. Therefore,

$$\begin{aligned}\|\mathrm{curl}(\boldsymbol{u}\circ\psi)\|_{H^{\mathrm{k}}(\mathcal{D})} &\leqslant \|(\mathrm{curl}\boldsymbol{u})\circ\psi\|_{H^{\mathrm{k}}(\mathcal{D})} + C\epsilon\|\boldsymbol{u}\circ\psi\|_{H^{\mathrm{k}+1}(\mathcal{D})}\\ &\leqslant C_1\|\mathrm{curl}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + (C_1 + C)\epsilon\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)}\,. \end{aligned} \tag{131a}$$

Similarly,

$$\|\mathrm{div}(\boldsymbol{u}\circ\psi)\|_{H^{\mathrm{k}}(\mathcal{D})} \leqslant C_1\|\mathrm{div}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + (C_1 + C)\epsilon\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)}\,. \tag{131b}$$

Let $\boldsymbol{n}$ be the outward-pointing unit normal to $\partial\mathcal{D}$. Then by the identity $\mathbf{N}\circ\psi = \dfrac{A^{\mathrm{T}}\boldsymbol{n}}{|A^{\mathrm{T}}\boldsymbol{n}|}$, we find that

$$\|(\mathbf{N}\circ\psi) - \boldsymbol{n}\|_{H^{\mathrm{k}-0.5}(\partial D)} \leqslant C_2(|\partial\mathcal{D}|_{H^{\mathrm{k}+0.5}})\epsilon\,.$$

Therefore, in addition to estimate (131a,b), we also have

$$\begin{aligned}\|\nabla_{\partial\mathcal{D}}(\boldsymbol{u}\circ\psi)\cdot\boldsymbol{n}\|_{H^{\mathrm{k}-0.5}(\partial\mathcal{D})} &\leqslant \|\nabla_{\partial\mathcal{D}}(\boldsymbol{u}\circ\psi)\cdot(\mathbf{N}\circ\psi)\|_{H^{\mathrm{k}-0.5}(\partial\mathcal{D})} + C_2\epsilon\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\partial\Omega)}\\ &\leqslant C_1(1+\epsilon)\|\nabla_{\partial\Omega}\boldsymbol{u}\cdot\mathbf{N}\|_{H^{\mathrm{k}-0.5}(\partial\Omega)} + C_2\epsilon\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)}\\ &\leqslant C_1\|\nabla_{\partial\Omega}\boldsymbol{u}\cdot\mathbf{N}\|_{H^{\mathrm{k}-0.5}(\partial\Omega)} + (C_1+C_2)\epsilon\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)}\,.\end{aligned}$$

Finally, by Theorem 1.2, there exists a generic constant $C_3 = C_3(|\partial\mathcal{D}|_{H^{\mathrm{k}+0.5}})$ such that

$$\|\boldsymbol{v}\|_{H^{\mathrm{k}+1}(\mathcal{D})} \leqslant C_3\Big[\|\boldsymbol{v}\|_{L^2(\mathcal{D})} + \|\mathrm{curl}\boldsymbol{v}\|_{H^{\mathrm{k}}(\mathcal{D})} + \|\mathrm{div}\boldsymbol{v}\|_{H^{\mathrm{k}}(\mathcal{D})} + \|\nabla_{\partial\mathcal{D}}\boldsymbol{v}\cdot\boldsymbol{n}\|_{H^{\mathrm{k}-0.5}(\partial\mathcal{D})}\Big] \quad \forall\, \boldsymbol{v}\in H^{\mathrm{k}+1}(\mathcal{D})\,.$$

Letting $\boldsymbol{v} = \boldsymbol{u}\circ\psi$, using (130) and (131) we find that

$$\begin{aligned}c_1(1-\epsilon)\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \leqslant C_3C_1\Big[&\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\mathrm{curl}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\mathrm{div}\boldsymbol{u}\|_{H^{\mathrm{k}}(\Omega)} + \|\nabla_{\partial\Omega}\boldsymbol{u}\cdot\mathbf{N}\|_{H^{\mathrm{k}-0.5}(\partial\Omega)}\Big]\\ &+ C_3(C_1+C_2+C)\epsilon\|\boldsymbol{u}\|_{H^{\mathrm{k}+1}(\Omega)} \qquad \forall\, \boldsymbol{u}\in H^{\mathrm{k}+1}(\Omega)\,.\end{aligned}$$

Since $\epsilon \ll 1$, the last term on the right-hand side can be absorbed by the left-hand side, yielding a linear inequality. The conclusion of Theorem 1.5 then follows by linear interpolation.

Appendix A. Proofs of the inequalities in Section 2.2

Proof of Proposition 2.4. We estimate $D^j f D^{\ell-j} g$ for $j = 1, \cdots, \ell-1$ as follows:

(1) If $1 \leqslant j \leqslant \frac{\mathrm{n}}{2}$, by the Sobolev inequalities

$$\|w\|_{L^{\frac{\mathrm{n}}{j-\epsilon}}(\mathrm{O})} \leqslant C_\epsilon\|w\|_{H^{\frac{\mathrm{n}}{2}-j+\epsilon}(\mathrm{O})} \quad (\text{if } 0<\epsilon<1)\,,$$
$$\|w\|_{L^{\frac{2\mathrm{n}}{\mathrm{n}-2(j-\epsilon)}}(\mathrm{O})} \leqslant C\|w\|_{H^{j-\epsilon}(\mathrm{O})}\,,$$

we find that

$$\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \leqslant \|D^j f\|_{L^{\frac{\mathrm{n}}{j-\epsilon}}(\mathrm{O})}\|D^{\ell-j} g\|_{L^{\frac{2\mathrm{n}}{\mathrm{n}-2(j-\epsilon)}}(\mathrm{O})} \leqslant C_\epsilon\|f\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})}\|g\|_{H^{\ell-\epsilon}(\mathrm{O})}\,.$$

(2) If $j = \ell$, by the Sobolev inequality

$$\|w\|_{L^\infty(\mathrm{O})} \leqslant C_\epsilon\|w\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})}\,,$$

we find that

$$\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \leqslant C_\epsilon\|f\|_{H^\ell(\mathrm{O})}\|g\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})}\,.$$

(3) If $\frac{\mathrm{n}}{2} < j < \ell$ (this happens only when $\frac{\mathrm{n}}{2} < \ell \leqslant \mathrm{k}$), we consider the following two sub-cases:

(a) *The case* $\ell \leqslant \mathrm{n}$: Similar to the previous case, by the Sobolev inequalities

$$\|w\|_{L^{\frac{2\mathrm{n}}{\mathrm{n}-2(\ell-j)}}(\mathrm{O})} \leqslant C\|w\|_{H^{\ell-j}(\mathrm{O})} \quad\text{and}\quad \|w\|_{L^{\frac{\mathrm{n}}{\ell-j}}(\mathrm{O})} \leqslant C\|w\|_{H^{\frac{\mathrm{n}}{2}-\ell+j}(\mathrm{O})}\,,$$

we obtain that

$$\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \leqslant \|D^j f\|_{L^{\frac{2\mathrm{n}}{\mathrm{n}-2(\ell-j)}}(\mathrm{O})}\|D^{\ell-j} g\|_{L^{\frac{\mathrm{n}}{\ell-j}}(\mathrm{O})} \leqslant C\|f\|_{H^\ell(\mathrm{O})}\|g\|_{H^{\frac{\mathrm{n}}{2}}(\mathrm{O})}\,.$$

(b) *The case* $\mathrm{n} < \ell \leqslant \mathrm{k}$: If $j > \mathrm{k} - \frac{\mathrm{n}}{2}$, by the Sobolev inequalities

$$\|w\|_{L^{\frac{2\mathrm{n}}{\mathrm{n}-2(\mathrm{k}-j)}}(\mathrm{O})} \leqslant C\|w\|_{H^{\mathrm{k}-j}(\mathrm{O})} \quad\text{and}\quad \|w\|_{L^{\frac{\mathrm{n}}{\mathrm{k}-j}}(\mathrm{O})} \leqslant C\|w\|_{H^{\frac{\mathrm{n}}{2}-\mathrm{k}+j}(\mathrm{O})}\,,$$

we obtain that

$$\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \leqslant \|D^j f\|_{L^{\frac{2\mathrm{n}}{\mathrm{n}-2(\mathrm{k}-j)}}(\mathrm{O})}\|D^{\ell-j} g\|_{L^{\frac{\mathrm{n}}{\mathrm{k}-j}}(\mathrm{O})} \leqslant C\|f\|_{H^{\mathrm{k}}(\mathrm{O})}\|g\|_{H^{\frac{\mathrm{n}}{2}-\mathrm{k}+\ell}(\mathrm{O})}\,.$$

Now suppose that $\frac{\mathrm{n}}{2} < j \leqslant \mathrm{k} - \frac{\mathrm{n}}{2}$. Note that if $0 < \epsilon < \frac{1}{2}$,

$$\|w\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})} \leqslant C_\epsilon \|w\|_{W^{j,\infty}(\mathrm{O})} \leqslant C_\epsilon \|w\|_{H^{\mathrm{k}}(\mathrm{O})}\,,$$

$$\|w\|_{H^{\frac{\mathrm{n}}{2}-\mathrm{k}+\ell}(\mathrm{O})} \leqslant C \|w\|_{H^{\ell-j}(\mathrm{O})} \leqslant C \|w\|_{H^{\ell-\epsilon}(\mathrm{O})}\,.$$

Therefore, by interpolation we obtain that

$$\begin{aligned}\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} &\leqslant \|f\|_{W^{j,\infty}(\mathrm{O})} \|g\|_{H^{\ell-j}(\mathrm{O})} \\ &\leqslant C_\epsilon \|f\|^{1-\alpha_j}_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})} \|f\|^{\alpha_j}_{H^{\mathrm{k}}(\mathrm{O})} \|g\|^{\alpha_j}_{H^{\frac{\mathrm{n}}{2}-\mathrm{k}+\ell}(\mathrm{O})} \|g\|^{1-\alpha_j}_{H^{\ell-\epsilon}(\mathrm{O})}\end{aligned}$$

for some $\alpha_j \in (0,1)$; thus Young's inequality implies that

$$\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \leqslant C_\epsilon \Big[\|f\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})} \|g\|_{H^{\ell-\epsilon}(\mathrm{O})} + \|f\|_{H^{\mathrm{k}}(\mathrm{O})} \|g\|_{H^{\frac{\mathrm{n}}{2}-\mathrm{k}+\ell}(\mathrm{O})} \Big].$$

Summing over all the possible ℓ, we conclude that for $0 < \epsilon < \frac{1}{2}$,

$$\sum_{j=1}^{\ell} \|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \leqslant \begin{cases} C_\epsilon \|f\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})} \|g\|_{H^{\ell-\epsilon}(\mathrm{O})} & \text{if } \ell \leqslant \frac{\mathrm{n}}{2}\,, \\ C_\epsilon \Big[\|f\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})} \|g\|_{H^{\ell-\epsilon}(\mathrm{O})} + \|f\|_{H^{\mathrm{k}}(\mathrm{O})} \|g\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})} \Big] & \text{otherwise}\,. \end{cases}$$

Estimate (11) is then concluded by the fact that for all $\epsilon \in \big(0, \frac{1}{4}\big)$,

$$\frac{\mathrm{n}}{2} + \epsilon \leqslant \mathrm{k} \quad \text{and} \quad \frac{\mathrm{n}}{2} + \epsilon \leqslant \ell - \epsilon \ \text{ if (in addition) } \ell > \frac{\mathrm{n}}{2}\,.$$

Finally, we conclude estimate (12) by an additional estimate

$$\|f D^\ell g\|_{L^2(\mathrm{O})} \leqslant \|f\|_{L^\infty(\mathrm{O})} \|g\|_{H^\ell(\mathrm{O})} \leqslant C \|f\|_{H^{\mathrm{k}}(\mathrm{O})} \|g\|_{H^\ell(\mathrm{O})}\,.$$ □

Proof of Corollary 2.6. By the definition of determinant and (12), it is easy to see that

$$\|J\|_{H^{\mathrm{k}}(\mathrm{O})} \leqslant C \|\nabla\psi\|^{\mathrm{n}}_{H^{\mathrm{k}}(\mathrm{O})}\,.$$

By Sobolev embedding $H^{\mathrm{k}}(\mathrm{O}) \subseteq \mathscr{C}^{0,\alpha}(\mathrm{O})$, we find that J is uniformly continuous on $\overline{\mathrm{O}}$. Since $J \neq 0$ in $\overline{\mathrm{O}}$ (by the virtue of that ψ being a diffeomorphism), $\|J\|_{L^\infty(\mathrm{O})} \geqslant \delta > 0$ for some δ (depending on J). Using the cofactor formula of the inverse of matrices, we find that

$$\|A\|_{L^2(\mathrm{O})} \leqslant \frac{1}{\delta} \|JA\|_{L^2(\mathrm{O})} \leqslant \frac{C}{\delta} \|\nabla\psi\|^{\mathrm{n}-2}_{H^{\mathrm{k}}(\mathrm{O})} \|\nabla\psi\|_{L^2(\mathrm{O})}\,. \tag{132}$$

Therefore, by interpolation and Young's inequality, with the help of (11) we find that

$$\begin{aligned}\|D^{\mathrm{k}} A\|_{L^2(\mathrm{O})} &\leqslant \frac{1}{\delta} \|J D^{\mathrm{k}} A\|_{L^2(\mathrm{O})} \leqslant \frac{1}{\delta} \|D^k (JA)\|_{L^2(\mathrm{O})} + \frac{1}{\delta} \sum_{j=1}^{\mathrm{k}} \binom{k}{j} \|D^j J D^{\mathrm{k}-j} A\|_{L^2(\mathrm{O})} \\ &\leqslant C_\delta \|\nabla\psi\|^{\mathrm{n}-1}_{H^{\mathrm{k}}(\mathrm{O})} + C_\delta \|J\|_{H^{\mathrm{k}}(\mathrm{O})} \|A\|_{H^{\mathrm{k}-\epsilon}(\mathrm{O})} \\ &\leqslant C_\delta \|\nabla\psi\|^{\mathrm{n}-1}_{H^{\mathrm{k}}(\mathrm{O})} + C_\delta \|J\|_{H^{\mathrm{k}}(\mathrm{O})} \|A\|^{1-\frac{\mathrm{k}}{\epsilon}}_{H^{\mathrm{k}}(\mathrm{O})} \|A\|^{\frac{\mathrm{k}}{\epsilon}}_{L^2(\mathrm{O})} \\ &\leqslant C_{\delta,\delta_1} \big(\|\nabla\psi\|_{H^{\mathrm{k}}(\mathrm{O})} \big) + \delta_1 \|A\|_{H^{\mathrm{k}}(\mathrm{O})}\,.\end{aligned}$$

Combining the estimate above with (132), by choosing $\delta_1 > 0$ small enough we conclude (15). □

Proof of Corollary 2.7. We prove (16) by induction. Define $J = \det(\nabla\psi)$ and $A = (\nabla\psi)^{-1}$. With the help of (15), the case that $\ell = 0$ is concluded by

$$\|f\|^2_{L^2(\Omega)} = \int_{\mathrm{O}} |(f\circ\psi)(y)|^2 J(y)\, dy \leqslant C(\|\nabla\psi\|_{H^{\mathrm{k}}(\mathrm{O})}) \|f\circ\psi\|^2_{L^2(\mathrm{O})} \tag{133}$$

and

$$\|f\circ\psi\|^2_{L^2(\mathcal{O})} = \int_\Omega |f(x)|^2 \frac{1}{(J\circ\psi^{-1})(x)}\,dx \leqslant \frac{1}{\delta}\|f\|^2_{L^2(\Omega)}\,, \tag{134}$$

where $\delta > 0$ is a lower bound for $\|J\|_{L^\infty(\mathcal{O})}$. Suppose that (16) holds for $\ell = j$. Then for $\ell = j+1$, by (12) and (15) we obtain that

$$\begin{aligned}\|D^{j+1}f\|_{L^2(\Omega)} &\leqslant \|Df\|_{H^j(\Omega)} \leqslant C(\|\nabla\psi\|_{H^{\mathrm{k}}(\mathcal{O})})\|(Df)\circ\psi\|_{H^j(\mathcal{O})}\\ &\leqslant C(\|\nabla\psi\|_{H^{\mathrm{k}}(\mathcal{O})})\|A^{\mathrm{T}}D(f\circ\psi)\|_{H^j(\mathcal{O})}\\ &\leqslant C(\|\nabla\psi\|_{H^{\mathrm{k}}(\mathcal{O})})\|A\|_{H^{\mathrm{k}}(\mathcal{O})}\|D(f\circ\psi)\|_{H^j(\mathcal{O})}\\ &\leqslant C(\|\nabla\psi\|_{H^{\mathrm{k}}(\mathcal{O})})\|f\circ\psi\|_{H^{j+1}(\mathcal{O})}\end{aligned}$$

and

$$\begin{aligned}\|D^{j+1}(f\circ\psi)\|_{L^2(\mathcal{O})} &= \big\|D^j\big[(Df)\circ\psi D\psi\big]\big\|_{L^2(\mathcal{O})} \leqslant \big\|(Df)\circ\psi D\psi\big\|_{H^j(\mathcal{O})}\\ &\leqslant C\|\nabla\psi\|_{H^{\mathrm{k}}(\mathcal{O})}\|(Df)\circ\psi\|_{H^j(\mathcal{O})}\\ &\leqslant C\|\nabla\psi\|_{H^{\mathrm{k}}(\mathcal{O})}\|Df\|_{H^j(\Omega)} \leqslant C\|\nabla\psi\|_{H^{\mathrm{k}}(\mathcal{O})}\|f\|_{H^{j+1}(\Omega)}\,,\end{aligned}$$

which, together with the (133) and (134), concludes the case that $\ell = j+1$. □

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