管院微積分聯合教學　會考一　**Oct 18, 2016**

考試時間 120 分鐘，題目卷為兩張紙，共四頁，滿分 120 分。所有題目的答案都請依題號順序依序寫在答案卷上，而是非與填充題必須寫在第一頁。答案卷務必寫學號、姓名，題目卷不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘內不得離場。考試期間禁止使用字典、計算機、任何通訊器材並請勿自行攜帶任何紙張，違者成績以零分計算，監試人員不得回答任何關於試題的疑問。**Questions are to be answered on the answer sheet provided.**

是非題 **True or False** (*20* points)，請答 **T** (True) 或 **F** (False)。每題 2 分。 (不需詳列過程，請依題號順序依序寫在答案卷第一頁上。)

F **1.** Let f and g be two functions, then $\frac{d}{dx}f(g(x)) = \frac{d}{dx}g(f(x))$.

In general, $f(g(x)) \neq g(f(x))$. Therefore $\frac{d}{dx}f(g(x))$ may not be $\frac{d}{dx}g(f(x))$.

T **2.** In the xy-plane, every linear equation represents a straight line.

This is part of Theorem 1 on page 40.

F **3.** If f is a continuous function on an open interval (a, b) and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).

Consider the function $f(x) = \begin{cases} -1, & \text{if } x = -1 \\ x+2, & \text{if } x \in (-1, 1] \end{cases}$

Then $f(-1)f(1) = -1 \cdot 3 < 0$, however $f(x) > 0$ on $(-1, 1)$.

F **4.** If the demand is elastic at p, then an increase in the unit price will cause the revenue to increase.

See the block on page 204.

T **5.** We say that f is continuous on the interval (a, b) if f is continuous at every point in (a, b).

See the definition on page 120.

F **6.** Consider the equation: $x^2 + y^2 = 1$. Then y is a function of x.

For a given $x = \frac{1}{\sqrt{2}}$, we obtain y is $-\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}$. So y is not a function of x (see definition on page 50). Or you can see this by vertical line test (on page 56).

7. Suppose that L_1 and L_2 are two lines in the plane with slope $m_1 \neq 0$ and $m_2 \neq 0$ respectively. If $m_1 \neq m_2$, then L_1 and L_2 intersect exactly one point.

Ans. (T)

Sol.: Let the equation of L_1 and L_2 respectively be:

$$(*) \quad \begin{cases} y = m_1 x + c_1 \\ y = m_2 x + c_2 \end{cases}$$

Since $m_1 \neq 0$, $m_2 \neq 0$ and $m_1 \neq m_2$, $(*)$ has a only one sol'n $\left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2}\right)$.

8. If f is differentiable, then $\dfrac{d}{dx}\left[\dfrac{f(x)}{x^2+1}\right] = \dfrac{f'(x)}{2x}$.

Ans.: (F)

In fact, by Quotient Rule,

$$\frac{d}{dx}\left[\frac{f(x)}{x^2+1}\right] = \frac{f'(x)(x^2+1) - f(x)(x^2+1)'}{(x^2+1)^2} = \frac{f'(x)(x^2+1) - 2x \cdot f(x)}{(x^2+1)^2}$$

9. $\displaystyle\lim_{x \to \infty} \frac{(x^{50}+1)^2}{x^{100}+2} = 1.$

Ans: (T)

Sol.: $$\lim_{x \to \infty} \frac{(x^{50}+1)^2}{x^{100}+2} = \lim_{x \to \infty} \frac{\left(1 + \frac{1}{x^{50}}\right)^2}{1 + \frac{2}{x^{100}}} = \frac{\lim_{x \to \infty} \left(1 + \frac{1}{x^{50}}\right)^2}{\lim_{x \to \infty} 1 + \frac{2}{x^{100}}}$$

$$= 1.$$

10. Let $f(x) = \begin{cases} \dfrac{|x|}{x} & , x \neq 0 \\ k & , x = 0 \end{cases}$. We can choose some value k such that the function f is continuous at $x = 0$.

Ans.: (F)

Sol.: Since $\displaystyle\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} 1 = 1$,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} (-1) = -1$$

We can not choose k such that $\displaystyle\lim_{x \to 0} f(x)$ exists.

Thus f is discontinuous at $x=0$ for any value k.

填充題 **Short answer questions** (*40* points)，每題 5 分。

(不需詳列過程，僅將答案依題號順序依序寫在答案卷第一頁上即可。)

1. Find the domain of

$$f(x) = \frac{\sqrt{x-3}}{x^2-4}.$$

Answer: $[3, \infty)$.

The denominator should not be zero.

So $x^2-4 \neq 0$, hence $x \neq -2$ and $x \neq 2$.

$\sqrt{x-3}$ makes sense only when $x-3 \geq 0$,

i.e., $x \geq 3$.

Therefore the domain is $[3, \infty)$.

2. We define $f(x) = \begin{cases} x-2 & , x < 1; \\ x^2-2x & , 1 \leq x \leq 3; \\ 2 & , x > 3. \end{cases}$

Find all points that $f(x)$ is not continuous at. If there is no such point, please put NONE in your answer sheet. Answer: $x=3$.

Since $f(x)$ is continuous on $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$,

it suffices to consider the continuity at 1 and 3.

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x-2 = -1$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2-2x) = -1$

give that f is continuous at $x=1$.

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2-2x = 3$ and $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2 = 2$,

so $f(x)$ is not continuous at $x=3$.

3. Consider the demand equation $f(p) = (1+p)^{-1}$, $p \geq 0$. Find the elasticity of demand at price $p = 20$. Answer: $\frac{20}{21}$.

$$E(P) = -\frac{pf'(p)}{f(p)} = -\frac{p(-1)(1+p)^{-2}}{(1+p)^{-1}} = p(1+p)^{-1}$$

$$\text{So } E(20) = 20(1+20)^{-1} = \frac{20}{21}$$

4. Find the value of

$$\lim_{x\to\infty} \frac{3x^{9/2} + 1000x^4}{x^5 + x^4 + 10000}.$$

Answer: 0.

$$\lim_{x\to\infty} \frac{3x^{\frac{9}{2}} + 1000x^4}{x^5 + x^4 + 10000}$$

$$= \lim_{x\to\infty} \frac{\frac{3x^{\frac{9}{2}} + 1000x^4}{x^5}}{\frac{x^5 + x^4 + 10000}{x^5}}$$

$$= \lim_{x\to\infty} \frac{3x^{-\frac{1}{2}} + 1000x^{-1}}{1 + x^{-1} + 10000x^{-5}}$$

$$= \frac{\lim_{x\to\infty} 3x^{-\frac{1}{2}} + 1000x^{-1}}{\lim_{x\to\infty} 1 + x^{-1} + 10000x^{-5}}$$

$$= \frac{0}{1}$$

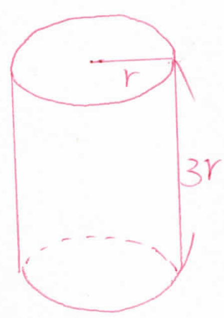
$$= 0$$

5. David wanna paint a right circular cylinder with radius r meters and height $3r$ meters. It costs 10 dollars to paint 1 square meter of an area. How much money does it cost to paint all the surface of the right circular cylinder? Answer: $80\pi r^2$.

The surface of the right circular cylinder

is $\pi r^2 + \pi r^2 + 2\pi r \cdot 3r = 8\pi r^2 \ m^2$

So the cost is $8\pi r^2 \cdot 10 = 80\pi r^2$



6. Let $f(x) = \begin{cases} k(x+1)^2 & , x \leq 0 \\ x+2 & , x > 0 \end{cases}$. Find the value of k that will make f continuous on $(-\infty, \infty)$. Answer: 2 .

Sol: It is easy to see that f is continuous at $x \neq 0$

Hence, if f is continuous at $x = 0$, then

we finaly have that f is continuous on $(-\infty, \infty)$.

This implies

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0) = k \cdot (0+1)^2 = k,$$

and so we obtain that

$$0 + 2 = 2 = k.$$

7. Let $F(x) = x^4 - 32x$. Find the point on the graph of F where the tangent line is horizontal. Answer: $(2, -48)$

Sol.: Let $(x_0, F(x_0))$ be the point on the graph of F where the tangent line is horizontal.

Then $F'(x_0) = 0 \Rightarrow 4x_0^3 - 32 = 0 \Rightarrow x_0 = 2$,

and so $(x_0, F(x_0)) = (2, -48)$.

8. Let $h(x) = \dfrac{f(x)g(x)}{f(x)+g(x)}$. If $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$ and $g'(1) = 4$, then find $h'(1)$. Answer:__________.

Sol.: By the Quotient and Product Rules, we obtain

$$h'(x) = \frac{(f(x)\cdot g(x))' \cdot (f(x)+g(x)) - (f(x)g(x))\cdot (f(x)+g(x))'}{(f(x)+g(x))^2}$$

$$= \frac{(f'(x)g(x)+f(x)g'(x))\cdot(f(x)+g(x)) - f(x)g(x)(f'(x)+g'(x))}{(f(x)+g(x))^2}.$$

We easily have

$$h'(1) = \frac{(3\cdot 2+1\cdot 4)(1+2) - 1\cdot 2\cdot(3+4)}{(1+2)^2} = \frac{16}{9}.$$

計算問答證明題 **Please show all your work** (*60* points)，每題 10 分，請依題號順序依序寫在答案卷上，可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. (*10* points) Find the derivatives of the following functions.

a. $f(x) = \dfrac{\sqrt{2x+1}}{x^2-1}$

b. $g(x) = (3x+1)^4(x^2-x+1)^3$

Sol.: a. By the Quotient and Chain Rules, we obtain

$$f'(x) = \frac{\left[(2x+1)^{\frac{1}{2}}\right]' \cdot (x^2-1) - (2x+1)^{\frac{1}{2}} (x^2-1)'}{(x^2-1)^2}$$

$$= \frac{\left[\frac{1}{2}(2x+1)^{-\frac{1}{2}} \cdot 2\right] \cdot (x^2-1) - (2x+1)^{\frac{1}{2}} \cdot 2x}{(x^2-1)^2}$$

$$= \frac{\frac{(x^2-1)}{\sqrt{2x+1}} - 2x\sqrt{2x+1}}{(x^2-1)^2} = \frac{-3x^2-2x-1}{(2x+1)^{\frac{1}{2}}(x^2-1)^2}.$$

b. By $\begin{cases} \text{Product Rule} \\ \text{Chain Rule} \end{cases}$ we easily obtain

$$g'(x) = 12(3x+1)^3(x^2-x+1)^3 + (3x+1)^4 \cdot 3(x^2-x+1)^2 \cdot (2x-1)$$

$$= 3(3x+1)^3(x^2-x+1)^2 \cdot \left[4(x^2-x+1) + (3x+1)(2x-1)\right]$$

$$= 3(3x+1)^3(x^2-x+1)^2 \cdot (10x^2-5x+3).$$

2. (*10* points) The quantity demanded per month, x, of a certain make of tablet PC is related to the average unit price, p (in dollars), of tablet PCs by the equation

$$x = f(p) = \frac{100}{9}\sqrt{810000 - p^2}$$

It is estimated that t months from now, the average price of a tablet PC will be given by

$$p(t) = \frac{400}{1 + \frac{1}{8}\sqrt{t}} + 200 \quad (0 \le t \le 60)$$

dollars. Find the rate at which the quantity demanded per month of the tablet PCs will be changing 16 months from now.

Sol: By the assumptions, we easily have

$x = f(p(t)) \quad (0 \le t \le 60)$,

and so the rate at which the quantity demanded per month is

$$\frac{dx}{dt} = f'(p(t)) \cdot p'(t) \quad (0 \le t \le 60)$$

$$= \frac{100}{9}\frac{1}{2}(810000 - p^2(t))^{-\frac{1}{2}}(-2p(t)) \cdot p'(t)$$

$$= -\frac{100}{9}(810000 - p^2(t))^{-\frac{1}{2}} \cdot p(t) \cdot \left(\frac{-400}{(1+\frac{1}{8}\sqrt{t})^2}\right) \cdot \frac{1}{8} \cdot \left(\frac{1}{2\sqrt{t}}\right)$$

$$= \frac{2500\,p(t)}{9\sqrt{t}\sqrt{810000 - p(t)^2}\,(1 + \frac{1}{8}\sqrt{t})^2}$$

When $t = 16$, $p(16) = \frac{400}{1+\frac{1}{8}\sqrt{16}} + 200 = \frac{1400}{3}$

This implies

$$\frac{dx}{dt}(16) = \frac{2500\left(\frac{1400}{3}\right)}{9\sqrt{16}\sqrt{810000 - \left(\frac{1400}{3}\right)^2} \cdot (1 + \frac{1}{8}\sqrt{16})^2}$$

$$\left(\text{or } \frac{3500000}{243\sqrt{810000 - \left(\frac{1400}{3}\right)^2}}\right)$$

3. (*10* points) The demand function for a product is given by

$$d = x^2 - 16x + 60 \qquad (0 \leq x \leq 6)$$

and the corresponding supply function is given by

$$s = x^2 + 3x + 3 \qquad (0 \leq x \leq 6)$$

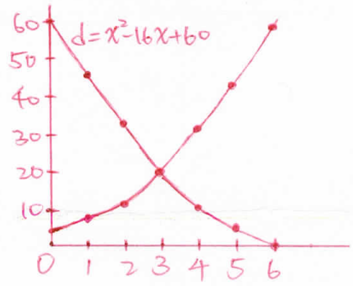
where d and s are in **dollars** and ~~s~~ x is measured in units of a **thousand**.

a. Sketch the graphs of those functions in **ONE** Cartesian coordinate system (4 points).

b. Find the equilibrium quantity and price (6 points).

a, $d = x^2 - 16x + 60 = x^2 - 16x + 64 - 4 = (x-8)^2 - 4$

$S = x^2 + 3x + 3 = x^2 + 3x + \frac{9}{4} + \frac{3}{4} = (x + \frac{3}{2})^2 + \frac{3}{4}$



b, Solve $x^2 - 16x + 60 = x^2 + 3x + 3$

Then $19x = 57$

So $x = 3$

Therefore the equilibrium quantity is 3,000 and the equilibrium price is $3^2 - 16 \cdot 3 + 60 = 21$.

4. (*10* points) Find the value of

$$\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{\sqrt{1-h} - 1.}$$

You may lose points if you explain things badly.

$$\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{\sqrt{1-h} - 1}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{\sqrt{1-h} - 1} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1-h} + 1} \cdot \frac{\sqrt{1-h} + 1}{\sqrt{1+h} + 1}$$

$$= \lim_{h \to 0} \frac{1+h-1}{1-h-1} \cdot \frac{\sqrt{1-h} + 1}{\sqrt{1+h} + 1}$$

$$= -\lim_{h \to 0} \frac{\sqrt{1-h} + 1}{\sqrt{1+h} + 1}$$

$$= -1$$

5. (*10* points) Suppose that a manufacturer of laptops has a monthly fixed cost 20,000 dollars and a variable cost $500x + 200\sqrt{x}$ dollars where x denotes the quantity.

a. Find the total monthly cost function $C(x)$ (2 points).

b. Find average cost function $\bar{C}(x)$ and marginal average cost function (4 points).

c. Evaluate $\lim_{x\to\infty} \bar{C}(x)$ and interpret your result (4 points).

a.
The total monthy cost function

$C(x) = 20{,}000 + 500x + 200\sqrt{x}$

b.
The averge cost function

$\bar{C}(x) = \frac{C(x)}{x} = \frac{20000}{x} + 500 + 200\frac{1}{\sqrt{x}}$

The marginal average cost function

$\bar{C}'(x) = -20000x^{-2} - 100x^{-\frac{3}{2}}$

c.
$\lim_{x\to\infty} \bar{C}(x) = \lim_{x\to\infty} 20000x^{-1} + 500 + 200x^{-\frac{1}{2}} = 500$

When we produce more, the cost of each laptop is close to 500 dollars.

6. (*10* points) The concentration of a certain drug in a patient's bloodstream t hours after injection is given by

$$C(t) = \frac{t}{t^2+2}$$

(a) Find the rate at which the concentration of the drug is changing with respect to time (2 points).

(b) How fast is the concentration changing 1 hour and 3 hours after the injection (4 points)?

(c) Evaluate $\lim_{t\to\infty} C(t)$, interpret your result (4 points).

(a)

$$C'(t) = \frac{d}{dt}\left(\frac{t}{t^2+2}\right) = \frac{(t^2+2)\cdot 1 - t\cdot 2t}{(t^2+2)^2} = \frac{2-t^2}{(t^2+2)^2}$$

(b)

This is to compute $C'(1)$ and $C'(3)$.

$$C'(1) = \frac{2-1^2}{(1^2+2)^2} = \frac{1}{9}$$

$$C'(3) = \frac{2-3^2}{(3^2+2)^2} = \frac{-7}{121}$$

(c)

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{t}{t^2+2} = \lim_{t\to\infty} \frac{\frac{t}{t^2}}{\frac{t^2}{t^2}+\frac{2}{t^2}} = 0$$

When time goes by, the concentration of that drug in patient's bloodstream will vanish.

(試題結束)