管院微積分聯合教學　會考二　Nov 29, 2016

考試時間 120 分鐘，題目卷為兩張紙，共三頁，滿分 120 分。所有題目的答案都請依題號順序依序寫在答案卷上，而是非與填充題必須寫在第一頁。答案卷務必寫學號、姓名，題目卷不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘內不得離場。考試期間禁止使用字典、計算機、任何通訊器材並請勿自行攜帶任何紙張，違者成績以零分計算，監試人員不得回答任何關於試題的疑問。**Questions are to be answered on the answer sheet provided.**

是非題 **True or False** (*20* points)，請答 **T** (True) 或 **F** (False)。每題 2 分。 (不需詳列過程，請依題號順序依序寫在答案卷第一頁上。)

F **1.** If $h(x) = f(x^2)$, then $h''(x) = (2x)^2 f''(x^2)$.

$h'(x) = f'(x^2)(2x)$,　$h''(x) = 2f'(x^2) + (2x)^2 f''(x^2)$.

F ~~T~~ **2.** If $A = f(\sqrt{x})$, then the percentage change in A is approximately equal to $\dfrac{50f'(\sqrt{x})}{\sqrt{x}f(\sqrt{x})}$.

$100\dfrac{\Delta A}{A} \approx 100\dfrac{dA}{A} = 100\dfrac{f'(\sqrt{x})\frac{1}{2\sqrt{x}}dx}{f(\sqrt{x})} = \dfrac{50f'(\sqrt{x})}{\sqrt{x}f(\sqrt{x})}dx$.

F **3.** The relative minimum of f is always less than or equal to the relative maximum of f.

Let $f(x) = x + \frac{1}{x}$. Then $f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$ and $x = -1, 1$ are critical number.

The sign chart of f' is $\underset{-1}{+} \quad \underset{0}{-} \quad \underset{1}{-} \quad +$

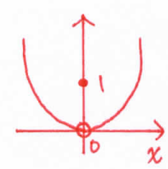
We have relative maximum $f(-1) = -2$ and relative minimum $f(1) = 2$. But $f(1) > f(-1)$.

F **4.** If $x = c$ is a critical number of f and $f'(x)$ changes sign from negative to positive as we move across $x = c$, then f has a relative minimum at $x = c$.

Let $f(x) = \begin{cases} x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Then $x=0$ is a critical number and

$f'(x) = 2x < 0,\ x < 0$; $f'(x) = 2x > 0,\ x > 0$.

But $f(0) = 1$ is not a relative minimum, as show in the figure.



T **5.** If $x < y$, then $\left(\frac{1}{e}\right)^x > \left(\frac{1}{e}\right)^y$.

Base $\frac{1}{e} < 1$, $(\frac{1}{e})^x > (\frac{1}{e})^y$ for $x < y$.

F **6.** Suppose $P(t)$ represents the population of bacteria at time t, and suppose $P'(t) > 0$ and $P''(t) < 0$; then the population is increasing at an increasing rate.

$P'(t) > 0 \Rightarrow P(t)$ is increasing

$P''(t) < 0 \Rightarrow P'(t)$ is decreasing where $P'(t)$ is the rate of change of $P(t)$ with respect to time.

So, $P(t)$ is increasing at a <u>decreasing</u> rate.

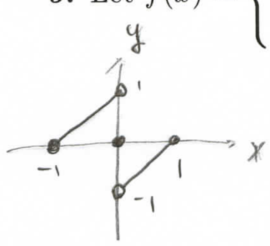
T **7.** If c is a critical number of f where $a < c < b$ and $f''(x) < 0$ on (a, b), then f has relative maximum at $x = c$.

$f''(x) < 0$ on (a, b) implies that f'' is defined on (a, b) hence $f'(x)$ is defined and continuous on (a, b). Then $f'(c) = 0$ if $c \in (a, b)$ is a critical point of f.
Also, $f''(c) < 0$. since $c \in (a, b)$. So by 2nd Derivative Test, f has a local maximum at $x = c$

T **8.** If $(c, f(c))$ is an infection point of f and if f' is defined and continuous at $x = c$, then $f'(c)$ is a relative extremum of f'.

That $(c, f(c))$ is an inflection point of f implies that $f''(x)$ is defined near c (except perhaps at $x=c$) and $f''(x) = (f')'(x)$ changes sign at $x=c$. Since f' is defined and continuous at $x=c$, by First derivative test $f'(c)$ is a local extremum of f'.

F **9.** Let $f(x) = \begin{cases} 1+x & , x \in [-1, 0) \\ 0 & , x = 0 \\ x-1 & , x \in (0, 1] \end{cases}$, then f has absolute extrema on $[-1, 1]$.



① $-1 \le f(x) \le 1$ on $[-1, 1]$

② f is strictly increasing on $[-1, 0)$, $\lim_{x \to 0^-} f(x) = 1$, but $f(x) \le 0$ on $[0, 1]$, so $f(x)$ has no absolute maximum on $[-1, 1]$.

③ f is strictly increasing on $(0, 1]$, $\lim_{x \to 0^+} f(x) = -1$. But $f(x) \ge 0$ on $[-1, 0]$, so $f(x)$ has no absolute minimum on $[-1, 1]$.

T **10.** $\ln(x^2 e^3) = 2 \ln |x| + 3$.

$$\begin{aligned} \ln(x^2 e^3) &= \ln x^2 + \ln e^3 \\ &= \ln |x|^2 + \ln e^3 \\ &= 2 \ln |x| + 3 \ln e \end{aligned}$$

填充題 **Short answer questions** (*40* points)，每題 5 分。

(不需詳列過程，僅將答案依題號順序依序寫在答案卷第一頁上即可。)

1. The volume of a right-circular cylinder of radius r and height h is $V = \pi r^2 h$. Suppose the radius and height of the cylinder are changing with respect to time t. At a certain instant of time, the radius and height of the cylinder are 2 and 4 in. and are increasing at the rate of 0.1 and 0.3 in./sec, respectively. How fast is the volume of the cylinder increasing? Answer:____________.

Differentials with respect to t,

$$\frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

Substituting $r=2$, $h=4$, $\frac{dr}{dt}=0.1$, and $\frac{dh}{dt}=0.3$, we have

$$\frac{dV}{dt} = 2\pi(2)(4)(0.1) + \pi(2)^2(0.3)$$
$$= (1.6+1.2)\pi = 2.8\pi$$

2. Find the absolute extrema, if any, of $f(x) = x - 3x^{1/3}$ on the interval $[-3, 27)$.

Answer:____________.

由 $f'(x) = 1 - x^{-2/3} = \frac{x^{2/3}-1}{x^{2/3}}$，得

臨界數 $x = -1, 0, 1$.

代入端點及臨界數

找 $x=27$ 的左極限，得

$f(-3) = -3 + 3^{4/3} = 3(3^{1/3}-1) \approx 1.33$

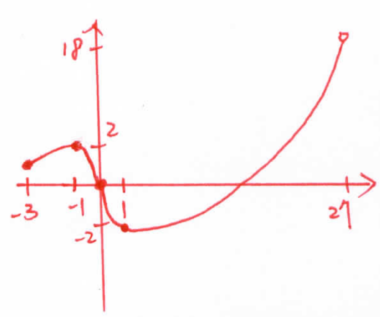
$f(-1) = -1 + 3 = 2$

$f(0) = 0$

$f(1) = 1 - 3 = -2$

$\lim_{x\to 27^-} f(x) = 27 - 3(3) = 18$

比較得絕對最小值 -2，無絕對最大值，如圖



3. An apple orchard has an average yield of 36 bushels of apples/tree if the tree density is 24 trees/acre. For each unit increase in tree density, the yield decreases by 2 bushels/tree. How many trees should be planted in order to maximize the yield? Answer:__________.

設 x 為超出的種植數量，得產量

$f(x) = (24+x)(36-2x) = -2x^2-12x+(24)(36)$

微分，$f'(x) = -4x-12$ 且臨界數 $x=-3$

又 $f''(x) = -4 < 0$，下凹，得 $x=-3$，即需種植 $24-3=21$ 棵時，可得最大產量.

4. Consider the equation $\sqrt{xy} = \frac{1}{4}x + y^2$. Use implicit differentiation to find the derivative dy/dx at $x = 4$, $y = 1$. Answer:__________.

$$\frac{d}{dx}(\sqrt{xy}) = \frac{d}{dx}\left(\frac{1}{4}x + y^2\right)$$

$$\Rightarrow \frac{1}{2}\cdot\frac{1}{\sqrt{xy}}\cdot\left(1\cdot y + x\cdot\frac{dy}{dx}\right) = \frac{1}{4} + 2y\cdot\frac{dy}{dx}$$

代入 $x=4$，$y=1$

$$\Rightarrow \frac{1}{2}\cdot\frac{1}{\sqrt{4}}\left(1\cdot 1 + 4\cdot\frac{dy}{dx}\right) = \frac{1}{4} + 2\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{4} + \frac{dy}{dx} = \frac{1}{4} + 2\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 2\cdot\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 0$$

5. Find the differential of the function $f(x) = (4x^2 - x)^{1/3}$.

Answer: $\left(\frac{8}{3}x - \frac{1}{3}\right)(4x^2 - x)^{\frac{-2}{3}}\,dx$

$$df = f'(x)\,dx$$

$$f'(x) = \left((4x^2-x)^{\frac{1}{3}}\right)' = \frac{1}{3}(4x^2-x)^{\frac{-2}{3}} \cdot (8x-1)$$

$$= \left(\frac{8}{3}x - \frac{1}{3}\right)(4x^2-x)^{\frac{-2}{3}}$$

$$\Rightarrow df = \left(\frac{8}{3}x - \frac{1}{3}\right)(4x^2-x)^{\frac{-2}{3}}\,dx$$

6. Find all horizontal asymptotes of the graph of the function $f(x) = \dfrac{|x|x}{x^2+1}$.

Answer: $y = \pm 1$ (or $y = -1$ and $y = 1$)

$$\lim_{x\to\infty} \frac{|x|\,x}{x^2+1} = \lim_{x\to-\infty} \frac{(-x)\,x}{x^2+1} = \lim_{x\to-\infty} \frac{-x^2/x^2}{(x^2+1)/x^2}$$

$$= \lim_{x\to-\infty} \frac{-1}{1+\frac{1}{x^2}} = -1$$

$$\lim_{x\to\infty} \frac{|x|\,x}{x^2+1} = \lim_{x^2+1} \frac{x\cdot x}{x^2+1} = \lim_{x\to\infty} \frac{x^2/x^2}{(x^2+1)/x^2}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = 1$$

$\Rightarrow$ Both $y=-1$ and $y=1$ are horizontal asymptotes of the graph of f

7. Let $f(x) = x^{1/3}$. Find the inflection point(s), if any, of f.

Answer: $x = 0$.

$f(x) = x^{\frac{1}{3}}$ is defined and continuous for all $x \in \mathbb{R}$

$f'(x) = \frac{1}{3} x^{\frac{-2}{3}}$ $\quad$ $f''(x) = \frac{-2}{9} x^{\frac{-5}{3}}$

$f''(x) > 0$ on $(-\infty, 0)$, $f''(x) < 0$ on $(0, \infty)$

$\Rightarrow$ $f''(x)$ changes sign at $x = 0$.

$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \frac{1}{3} \cdot x^{\frac{-2}{3}} = \infty$

$\Rightarrow$ the graph of f has a (vertical) tangent line at the point $(0, f(0)) = (0, 0)$

So, $x = 0$ is the inflection point of f

8. Find the value of k so that $2^x = e^{kx}$ for all x. Answer: $k = \ln 2$.

$2^x = e^{kx} = (e^k)^x$

$\Rightarrow$ $2 = e^k$

$\Rightarrow$ $\ln 2 = \ln e^k = k \ln e = k$

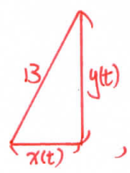
$\Rightarrow$ $k = \ln 2$

(下頁還有試題)

計算問答證明題 **Please show all your work** (*60* points)，每題 10 分，請依題號順序依序寫在答案卷上，可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. (*10* points) The base of a 13-ft ladder leaning against a wall begins to slide away from the wall. At the instant of time when the top is 12 ft from the ground, the base is moving at the rate of 8 ft/sec. How fast is the top of the ladder sliding down the wall at that instant of time?

如图



得 $x^2+y^2=13^2$ 且當 $y=12$, $\frac{dx}{dt}=8$, 問 $\frac{dy}{dx}$ 為何?

對 t 微分, 得 $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$, 即 $x\frac{dx}{dt}+y\frac{dy}{dt}=0$

代入 $y=12$, $\frac{dx}{dt}=8$ 以及 $x=\sqrt{169-144}=5$, 得

$5\cdot 8+12\frac{dy}{dt}=0$, 以及 $\frac{dy}{dt}=-\frac{40}{12}=-\frac{10}{3}$, 即頂端以 $10/3$ 呎/秒的速率下滑.

2. (*10* points) Let $f(x) = \dfrac{x-1}{x^2}$. Find the relative extrema, the inflection points, and the vertical and horizontal asymptotes, if any, of f. Then sketch the graph of f.

$f'(x) = \dfrac{x^2-(x-1)(2x)}{x^4} = \dfrac{2x-x^2}{x^4} = \dfrac{2-x}{x^3}$，臨界數 $x=2$

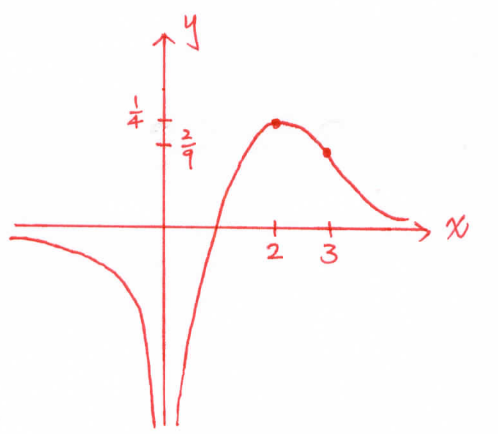
f': $\underset{0}{-}\ \underset{2}{+}\ -$，有相對極大值 $f(2)=\frac{1}{4}$

$f''(x) = \dfrac{-x^3-(2-x)(3x^2)}{x^6} = \dfrac{2x^3-6x^2}{x^6}$，反曲候選數 $x=3$

f'': $\underset{0}{-}\ \underset{3}{-}\ +$ 得反曲點 $(3, \frac{2}{9})$

$\lim_{x\to\pm\infty} f(x)=0$，得水平漸近線 $y=0$，

$\lim_{x\to 0^{\pm}} f(x) = -\infty$，得垂直漸近線 $x=0$.



3. (*10* points) The quantity demanded each month of the Sicard sports watch is related to the unit price by the equation $p = \dfrac{50}{0.01x^2+1}$, $0 \le x \le 20$, where p is measured in dollars and x is measured in units of a thousand. To yield a maximum revenue, how many watches must be sold?

$R(x) = px = \dfrac{50x}{0.01x^2+1}$, $0 \le x \le 20$

$R'(x) = 50\left[\dfrac{0.01x^2+1-0.02x^2}{(0.01x^2+1)^2}\right] = 50\,\dfrac{1-0.01x^2}{(0.01x^2+1)^2} = 0$, 得 $x=10$.

$R'(x)$: $0 \;\; + \;\; 10 \;\; - \;\; 20$ 當 $x=10$, 即銷售 10,000 隻時, 得最大收益

或比較 $R(0)=0$, $R(10) = \dfrac{500}{2} = 250$, $R(20) = \dfrac{1000}{5} = 200$, 得 $x=10$, 即銷售 10,000 隻時可獲最大收益.

4. (*10* points) Find $\frac{d^2y}{dx^2}$ if $xy - y^3 = 4$ in terms of x and y.

Find $\frac{dy}{dx}$:

$$\frac{d}{dx}(xy - y^3) = \frac{d}{dx}(4)$$

$$\Rightarrow 1\cdot y + x\cdot\frac{dy}{dx} - 3y^2\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow (3y^2 - x)\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{3y^2 - x}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{3y^2-x}\right)$$

$$= \frac{\frac{dy}{dx}\cdot(3y^2-x) - y\cdot\frac{d}{dx}(3y^2-x)}{(3y^2-x)^2}$$

$$= \frac{\frac{dy}{dx}\cdot(3y^2-x) - y\cdot\left(6y\cdot\frac{dy}{dx} - 1\right)}{(3y^2-x)^2}$$

$$= \frac{y - (3y^2+x)\cdot\frac{dy}{dx}}{(3y^2-x)^2}$$

$$= \frac{y - (3y^2+x)\cdot\frac{y}{3y^2-x}}{(3y^2-x)^2}$$

$$= \frac{y(3y^2-x) - y(3y^2+x)}{(3y^2-x)^3}$$

$$= \frac{-2xy}{(3y^2-x)^3}$$

Ans. $\boxed{\frac{-2xy}{(3y^2-x)^3}}$

5. (*10* points) Use a differential to approximate the quantity $(80.9)^{1/4}$.

Consider the function $f(x) = x^{\frac{1}{4}}$,

then $(80.9)^{\frac{1}{4}} = f(80.9)$

$80.9 \approx 81$ and $f(81) = 81^{\frac{1}{4}} = 3$

Take $x = 81$. $\Delta x = 80.9 - 81 = -0.1$

$f(x+\Delta x) - f(x) \approx f'(x)\,dx$, $dx = \Delta x$

$\Rightarrow f(80.9) - f(81) \approx f'(81) \cdot (-0.1)$

$f'(x) = \frac{1}{4} x^{\frac{-3}{4}}$

$f'(81) = \frac{1}{4} \cdot (81)^{\frac{-3}{4}} = \frac{1}{4} \cdot \frac{1}{27} = \frac{1}{108}$

$\Rightarrow f(80.9) \approx f(81) + \frac{1}{108}(-0.1)$

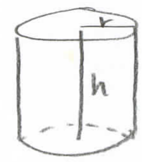
$\Rightarrow f(80.9) \approx 3 - \frac{1}{1080}$

$\Rightarrow \boxed{80.9^{\frac{1}{4}} \approx 3 - \frac{1}{1080}}$

6. (*10* points) Betty Moore company requires that its corned beaf hash containers have a capacity of 128 cubic inces, have the shape of a a right circular cylinder, and made of aluminum. Determine the radius and height of the container that requires the least amount of metal.

r: radius of the container

h: height of the container



$\Rightarrow$ Volume $= \pi r^2 h = 128$

$\Rightarrow h = \frac{128}{\pi r^2}$

The total amount of metal required is

$$2 \cdot \pi r^2 + 2\pi r \cdot h = 2\pi r^2 + 2\pi r \cdot \frac{128}{\pi r^2}$$

$$= 2\pi r^2 + \frac{256}{r}$$

Let $f(r) = 2\pi r^2 + \frac{256}{r}$, $r > 0$

$f'(r) = 4\pi r - \frac{256}{r^2}$, $r > 0$

$f'(r) = 0 \Rightarrow 4\pi r = \frac{256}{r^2} \Rightarrow r^3 = \frac{64}{\pi} \Rightarrow r = \frac{4}{\sqrt[3]{\pi}}$

$f''(r) = 4\pi + \frac{512}{r^3} > 0$ for all $r > 0$.

In particular, $f''\left(\frac{4}{\sqrt[3]{\pi}}\right) > 0 \Rightarrow f\left(\frac{4}{\sqrt[3]{\pi}}\right)$ is a relative minimum on $(0, \infty)$.

Since $f'' > 0$ on $(0, \infty)$ f attains its absolute minimum at $r = \frac{4}{\sqrt[3]{\pi}}$, and $h = \frac{128}{\pi \cdot \frac{16}{\pi^{\frac{2}{3}}}} = \frac{8}{\sqrt[3]{\pi}}$.

$$\Rightarrow \boxed{r = \frac{4}{\sqrt[3]{\pi}} \text{ (in.)}, \quad h = \frac{8}{\sqrt[3]{\pi}} \text{ (in.)}}$$

(試題結束)