管院微積分聯合教學　會考三　Jan 10, 2017

考試時間 120 分鐘，題目卷為兩張紙，共四頁，滿分 120 分。所有題目的答案都請依題號順序依序寫在答案卷上，而是非與填充題必須寫在第一頁。答案卷務必寫學號、姓名，題目卷不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘內不得離場。考試期間禁止使用字典、計算機、任何通訊器材並請勿自行攜帶任何紙張，違者成績以零分計算，監試人員不得回答任何關於試題的疑問。**Questions are to be answered on the answer sheet provided.**

是非題 **True or False** (*20* points)，請答 **T** (True) 或 **F** (False)。每題 2 分。(不需詳列過程，請依題號順序依序寫在答案卷第一頁上。)

F **1.** If f is a continuous function on an open interval (a, b) and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).

Consider the function $f(x) = \begin{cases} -1 & , \text{if } x=-1 \\ x+2 & , \text{if } x \in (-1, 1] \end{cases}$

Then $f(-1)f(1) = -1 \cdot 3 < 0$, however $f(x) > 0$ on $(-1, 1)$.

T **2.** If $x < y$, then $\left(\frac{1}{e}\right)^x > \left(\frac{1}{e}\right)^y$.

Base $\frac{1}{e} < 1$, $(\frac{1}{e})^x > (\frac{1}{e})^y$ for $x<y$.

T **3.** The effective interest rate $\hat{r}_{\text{eff}}$ that corresponds to a nominal interest rate r per year compounded continuously is given by $\hat{r}_{\text{eff}} = e^r - 1$.

根據定義，$\hat{r}_{eff}$ 滿足

$P(1+\hat{r}_{eff}) = Pe^r$，即 $1+\hat{r}_{eff} = e^r$，故 $\hat{r}_{eff} = e^r - 1$.

T **4.** If $x^2 + \ln y = 10$, then $\dfrac{dy}{dx} = -2xy$.

隱微分，得 $2x + \frac{1}{y}\frac{dy}{dx} = 0$，故 $\frac{dy}{dx} = -2xy$.

T **5.** $\displaystyle\int \left(\sqrt{x}e^x + \frac{2}{3}x^{3/2}e^x\right) dx = \frac{2}{3}x^{3/2}e^x + C.$

微分等式右边，得 $\frac{d}{dx}(\frac{2}{3}x^{3/2}e^x + C) = \frac{2}{3}\frac{3}{2}x^{1/2}e^x + \frac{2}{3}x^{3/2}e^x$

$= \sqrt{x}e^x + \frac{2}{3}x^{3/2}e^x$，

為等式左边的被積函數，

故根據不定積分的定義，$\int(\sqrt{x}e^x + \frac{2}{3}x^{3/2}e^x)dx = \frac{2}{3}x^{3/2}e^x + C$

F **6.** $\displaystyle\int \frac{1}{x^2+1}\, dx = \ln|x^2+1| + C.$

因為 $\frac{d}{dx}(\ln|x^2+1|) = \frac{1}{x^2+1}(2x) = \frac{2x}{x^2+1} \neq \frac{1}{x^2+1}$，

由不定積分的定義，$\int \frac{1}{x^2+1}dx \neq \ln|x^2+1| + C$.

事實上，$\int \frac{1}{x^2+1}dx = \tan^{-1}x + C$.

T **7.** If f is continuous on $[a, b]$ and $a < c < b$, then $\int_c^b f(x)\,dx = \int_c^a f(x)\,dx + \int_a^b f(x)\,dx$.

根據定積分的性質，$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$,

即 $\int_c^b f(x)\,dx = \int_a^b f(x)\,dx - \int_a^c f(x)\,dx$

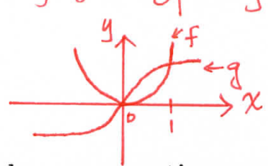
$= \int_a^b f(x)\,dx - [-\int_c^a f(x)\,dx]$

$= \int_c^a f(x)\,dx + \int_a^b f(x)\,dx$.

T **8.** The area of the region bounded by the graphs of the functions $f(x) = x^2$ and $g(x) = x^{1/3}$ is given by $\int_0^1 \left(x^{1/3} - x^2\right)\,dx$.

解 $x^2 = x^{1/3}$，得 $x^{1/3}(x^{5/3} - 1) = 0$，故 $x = 0, 1$.

又 $f(\frac{1}{8}) = \frac{1}{64} < g(\frac{1}{8}) = \frac{1}{2}$，故对所有在 $[0,1]$ 内的 x，$f(x) < g(x)$，如圖



因此，面積 $= \int_0^1 (x^{1/3} - x^2)\,dx$.

T **9.** If f and g are continuous on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then $\int_a^b [f(x) - g(x)]\,dx \geq 0$.

因為對所有 $[a,b]$ 内的 x，$[f(x) - g(x)] \geq 0$，得

$\int_a^b [f(x) - g(x)]\,dx \geq 0$.

T **10.** The consumers' surplus is given by $CS = \int_0^{\bar{x}} D(x)\,dx - \bar{p}\bar{x}$, where $D(x)$ is the demand function, $\bar{p}$ is the unit market price, and $\bar{x}$ is the quantity sold.

根據定義，$CS = \int_0^{\bar{x}} D(x)\,dx - \bar{p}\bar{x}$.

填充題 **Short answer questions** (40 points)，每題 5 分。

(不需詳列過程，僅將答案依題號順序依序寫在答案卷第一頁上即可。)

1. Let $h(x) = \dfrac{f(x)g(x)}{f(x)+g(x)}$. If $f(1)=3$, $g(1)=1$, $f'(1)=4$ and $g'(1)=2$, then find $h'(1)$. Answer:___________.

By the Quotient and Product Rules, we obtain

$$h'(x) = \frac{(f(x)\cdot g(x))'\cdot(f(x)+g(x)) - (f(x)g(x))\cdot(f(x)+g(x))'}{(f(x)+g(x))^2}$$

$$= \frac{(f'(x)g(x)+f(x)g'(x))\cdot(f(x)+g(x)) - f(x)g(x)(f'(x)+g'(x))}{(f(x)+g(x))^2}$$

We easily have

$$h'(x) = \frac{(4\cdot 1+3\cdot 2)(3+1) - 3\cdot 1\cdot(4+2)}{(3+1)^2}$$

$$= \frac{(4+6)\times 4 - 3\times 6}{4^2} = \frac{22}{16} = \frac{11}{8}$$

2. Find the derivative of the function $f(x) = \dfrac{2\ln x}{x}$. Answer:___________.

$$f'(x) = \frac{2\cdot\frac{1}{x}\cdot x - 2\ln x}{x^2} = \frac{2-2\ln x}{x^2}$$

3. Evaluate $\int_0^1 \frac{e^x}{1+e^x}\,dx$. Answer:___________.

Let $u = 1+e^x$, so $du = e^x dx$.

Then $\int_0^1 \frac{e^x}{1+e^x}\,dx = \int_2^{1+e} \frac{1}{u}\,du$

$= \ln|u| \Big|_2^{1+e}$

$= \ln(1+e) - \ln 2$

$\left(\text{or } \ln \frac{1+e}{2}\right)$

4. How long will it take \$5,000 to grow to \$15,000 if the investment earns interest at the rate of 4%/year compounded semiannually. Answer:___________.

根據複利公式，得 $15000 = 5000\left(1+\frac{0.04}{2}\right)^{2t}$

即 $3 = (1.02)^{2t}$

取 ln，得 $\ln 3 = 2t \ln 1.02$

因此，$t = \frac{\ln 3}{2\ln 1.02}$

5. Find the absoluate minimum of the function $f(x) = (1-2x)e^{-x}$ on $[0, \infty)$.

Answer: ___________.

由 $f'(x) = -2e^{-x} + (1-2x)(-e^{-x}) = (2x-3)e^{-x} = 0$，得臨界數 $x = \frac{3}{2}$

又 f': $\underset{0}{\vdash}\ \overset{-}{\text{———}}\ \underset{3/2}{+}\ \overset{+}{\text{———}}$,

得絕對最小值 $f(\frac{3}{2}) = -2e^{-\frac{3}{2}}$.

6. The daily marginal profit function associated with producing and selling TexaPep hot sauce is $P'(x) = -0.000006x^2 + 6$, where x denotes the number of cases (each case contains 24 bottles) produced and sold daily, and $P'(x)$ is measured in dollars per unit. The fixed cost is \$500. What is the total profit realizable from producing and selling 1000 cases of TexaPep per day?

Answer:___________.

$P(x) = \int (-0.000006x^2 + 6)\, dx$

$= -0.000002x^3 + 6x + C$

因為固定成本為\$500，得 $P(0) = R(0) - C(0) = 0 - 500 = -500$

代入，得 $-500 = 0 + C$，即 $C = -500$

因此，$P(x) = -0.000002x^3 + 6x - 500$

且 $P(1000) = -0.000002(1000)^3 + 6(1000) - 500$

$= -2000 + 6000 - 500 = 3500$.

7. Evaluate the definite integral $\int_{-1}^{1} \frac{x}{1+x^{2/3}}\,dx$. Answer:___________.

令 $f(x) = \frac{x}{1+x^{2/3}}$，得 $f(-x) = \frac{-x}{1+(-x)^{2/3}} = -\frac{x}{1+x^{2/3}} = -f(x)$. 即 f 為奇函數。因此，$\int_{-1}^{1} \frac{x}{1+x^{2/3}}\,dx = 0$

8. Aiso deposits \$150/month in a savings account paying 5%/year compounded continuously. Estimate the amount that will be in his account after 15 years. Answer:___________.

根據年金結餘公式 $A = \frac{mP}{r}(e^{rT}-1)$，得

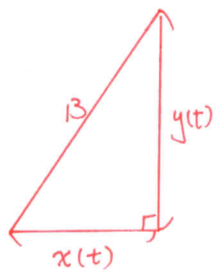
結餘為 $\frac{(12)(150)}{0.05}(e^{0.05(15)}-1)$

$= 36000(e^{0.75}-1)$

計算問答證明題 **Please show all your work** (*60* points)，每題 10 分，請依題號順序依序寫在答案卷上，可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. (*10* points) The base of a 13-ft ladder leaning against a wall begins to slide away from the wall. At the instant of time when the top is 12 ft from the ground, the base is moving at the rate of 8 ft/sec. How fast is the top of the ladder sliding down the wall at that instant of time?

如图



得 $x^2+y^2=13^2$ 且當 $y=12$, $\frac{dx}{dt}=8$, 問 $\frac{dy}{dt}$ 為何?

對 t 微分, 得 $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$, 即 $x\frac{dx}{dt}+y\frac{dy}{dt}=0$

代入 $y=12$, $\frac{dx}{dt}=8$ 以及 $x=\sqrt{169-144}=5$, 得

$5\cdot 8+12\frac{dy}{dt}=0$, 以及 $\frac{dy}{dt}=-\frac{40}{12}=-\frac{10}{3}$,

即頂端以 $10/3$ 呎/秒 的速率下滑.

2. (*10* points) The length (in centimeters) of a typical Pacific halibut t years old is approximately

$$f(t) = 200(1 - 0.956e^{-0.18t})$$

a. What is the length of a typical 5-year-old Pacific halibut?

b. How fast is the length of a typical 5-year-old Pacific halibut increasing?

c. What is the maximum length a typical Pacific halibut can attain?

a. $f(5) = 200(1 - 0.956e^{-0.9})$

b. $f'(t) = 200 \times (-0.956) \times (-0.18)\, e^{-0.18t}$

$f'(5) = 200 \times (-0.956) \times (-0.18)\, e^{-0.9}$

c. $\lim_{t \to \infty} f(t) = 200$ (cm)

3. (*10* points) The rate of change of the unit price p (in dollars) of Apex women's boots is given by

$$p'(x) = \frac{-250x}{(16+x^2)^{3/2}}$$

where x is the quantity demanded daily in units of a hundred. Find the demand function for these boots if the quantity demanded daily is 300 pair ($x = 3$) when the unit price is \$55/pair.

$p(x) = \int p'(x)\,dx = \int \frac{-250x}{(16+x^2)^{3/2}}\,dx$

Let $u = 16+x^2$, so $du = 2x\,dx$. Then

$p(x) = \int \frac{-250x}{u^{3/2}}\left(\frac{1}{2x}\,du\right)$

$= -125\int u^{-3/2}\,du$

$= -125\left(-2u^{-\frac{1}{2}}\right) + C$

$= 250\,u^{-\frac{1}{2}} + C = 250\,(16+x^2)^{-\frac{1}{2}} + C$

Use condition $p(3) = 55$,

$p(3) = 250\,(16+3^2)^{-\frac{1}{2}} + C = 50 + C$

So $C = 5$, thus $p(x) = 250\,(16+x^2)^{-\frac{1}{2}} + 5$

4. (*10* points) The production of oil (in millions of barrels per day) extracted from oil sands in Canada is projected to grow according to the function

$$P(t) = \frac{4.76}{1 + 4.11e^{-0.22t}}, \quad 0 \le t \le 20,$$

where t is measured in years, with $t = 0$ corresponding to 2005. What is the expected total production of oil from oil sands over the years from 2005 until 2025 (t=20)?

$$\int_0^{20} P(t)\,dt = \int_0^{20} \frac{4.76}{1+4.11e^{-0.22t}}\,dt$$

$$= \int_0^{20} \frac{4.76e^{0.22t}}{e^{0.22t}+4.11}\,dt \text{ . Put } I = \int \frac{4.76e^{0.22t}}{e^{0.22t}+4.11}\,dt$$

Let $u = e^{0.22t} + 4.11$, so $du = 0.22\, e^{0.22t}\,dt$

Then $I = \frac{4.76}{0.22} \int \frac{1}{u}\,du = \frac{4.76}{0.22} \ln|u| + C$

$$= \frac{4.76}{0.22} \cdot \ln\left(e^{0.22t} + 4.11\right) + C$$

So $\int_0^{20} \frac{4.76\, e^{0.22t}}{e^{0.22t}+4.11}\,dt = \left[\frac{4.76}{0.22} \ln\left(e^{0.22t} + 4.11\right)\right]_0^{20}$

$$= \frac{4.76}{0.22}\left[\ln(e^{4.4}+4.11) - \ln(1+4.11)\right]$$

5. (*10* points) The concentraction of a certain drug on a patient's bloodstream t hr after injection is $C(t) = \dfrac{0.2t}{t^2+1}$ mg/cm^3. Determine the average concentration of the drug in the patient's bloodstream over the first 5 hr after the drug is injected.

平均濃度 $= \frac{1}{5}\int_0^5 \frac{0.2t}{t^2+1}\,dt$

$= \frac{1}{5}(0.2)(\frac{1}{2})\int_0^5 \frac{1}{t^2+1}(2t)\,dt$

$= \frac{1}{50}\ln(t^2+1)\Big|_0^5 = \frac{1}{50}\ln 26$

6. (*10* points) In a study conducted by a certain country's Economic Development Board, it was found that the Lorenz curve for the distribution of income of stockbrokers was described by the function $f(x) = \frac{11}{12}x^2 + \frac{1}{12}x$ and that of high school teachers by the function $g(x) = \frac{5}{6}x^2 + \frac{1}{6}x$.

a. Compute the coefficient of inequality for each Lorenz curve.

b. Which profession has a more equitable income distribution?

a、針对股票經紀人，不等係數

$$L_1 = 2\int_0^1 [x - (\tfrac{11}{12}x^2 + \tfrac{1}{12}x)]\,dx$$
$$= 2\int_0^1 (\tfrac{11}{12}x - \tfrac{11}{12}x^2)\,dx$$
$$= \tfrac{11}{6}(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3)\Big|_0^1$$
$$= \tfrac{11}{6}(\tfrac{1}{2} - \tfrac{1}{3}) = \tfrac{11}{36}$$

針对高中老師，不等係數

$$L_2 = 2\int_0^1 [x - (\tfrac{5}{6}x^2 + \tfrac{1}{6}x)]\,dx$$
$$= 2\int_0^1 (\tfrac{5}{6}x - \tfrac{5}{6}x^2)\,dx$$
$$= \tfrac{5}{3}(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3)\Big|_0^1$$
$$= \tfrac{5}{3}(\tfrac{1}{2} - \tfrac{1}{3}) = \tfrac{5}{18}$$

b、因為 $L_1 = \frac{11}{36} > L_2 = \frac{5}{18} = \frac{10}{36}$，高中老師有較均等的收入分布.

(試題結束)