管院微積分聯合教學　會考五　**May 2, 2017**

考試時間 120 分鐘，題目卷為兩張紙，共四頁，滿分 120 分。所有題目的答案都請依題號順序依序寫在答案卷上，而是非與填充題必須寫在第一頁。答案卷務必寫學號、姓名，題目卷不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘內不得離場。考試期間禁止使用字典、計算機、任何通訊器材並請勿自行攜帶任何紙張，違者成績以零分計算，監試人員不得回答任何關於試題的疑問。**Questions are to be answered on the answer sheet provided.**

是非題 **True or False** (*20* points)，請答 **T** (True) 或 **F** (False)。每題 2 分。(不需詳列過程，請依題號順序依序寫在答案卷第一頁上。)

T

1. If $h(x,y) = f(x)g(y)$, where f is continuous on $[a,b]$ and g is continuous on $[c,d]$, then

$$\int\int_R h(x,y)\,dA = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)\,dy\right].$$

$$\iint_R f(x)g(y)\,dA = \int_c^d\left[\int_a^b f(x)g(y)\,dx\right]dy$$
$$= \int_c^d g(y)\left[\int_a^b f(x)\,dx\right]dy$$
$$= \left[\int_a^b f(x)\,dx\right]\int_c^d g(y)\,dy$$

T

2. The function $Q(t) = C$ is a solution to the differential equation

$$\frac{dQ}{dt} = kQ(Q-C)^2.$$

Let $Q(t) = C$, then

$$\frac{dQ}{dt} = \frac{dC}{dt} = 0 \quad \text{and}$$

$$kQ(Q-C)^2 = kQ\cdot(C-C)^2 = 0$$

$\Rightarrow Q(t) = C$ satisfies $\frac{dQ}{dt} = kQ(Q-C)^2$, hence is a sol'n.

F **3.** If f is a probability density function on $[a, b]$, then $0 \le f(x) \le 1$ on $[a, b]$.

f is a probabity density function on $[a,b]$
if $f(x) \ge 0$ on $[a,b]$ and $\int_a^b f(x)\,dx = 1$.
Then condition $0 \le f(x) \le 1$ does not true
for a general.

F **4.** The function $f(x) = \frac{3}{2}x^2 - 3x$ is a probability density function on $[1, 3]$.

$f(x) = \frac{3}{2}x(x-2) \ngeq 0$ on $[1,3]$

f is not a probabity density function on $[1,3]$

T **5.** If Z is the standard normal random variable, then $P(Z < -0.9) = P(Z > 0.9)$.

The graph of the probabity density function
$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ of Z is symmetric with respect to the
line $z = 0$, so $P(Z < -z) = P(Z > z)$. $\forall z$
In particular, $P(Z < -0.9) = P(Z > 0.9)$.

F **6.** The differential equation $y' = (x^2 + y)(e^{x+y})$ is separable.

Since (x^2+y) is not of the form $f(x)g(y)$,
$y' = (x^2+y)e^{x+y} = (x^2+y)e^x e^y$ is NOT separable.

T **7.** Suppose that X and Y has joint density function f on $(-\infty, \infty) \times (-\infty, \infty)$, then

$$P(a \leq X \leq b) = \int_a^b \int_{-\infty}^{\infty} f(x, y)\, dydx.$$

$$P(a \leq X \leq b) = P(a \leq X \leq b, -\infty < Y < \infty)$$
$$= \int_a^b \int_{-\infty}^{\infty} f(x, y)\, dy\, dx$$

F **8.** Suppose that X and Y have exponential density functions f and g with parameter λ and μ respectively. If $\lambda > \mu$, then $E[X] \geq E[Y]$.

The mean of an exponential random variable with parameter λ is $\frac{1}{\lambda}$.

So $E[X] = \frac{1}{\lambda} < \frac{1}{\mu} = E[Y]$.

F **9.** A normal distribution with mean μ and variance σ^2 is symmetric according to $x = 0$.

It is symmetric to $x = \mu$ instead of $x = 0$.

F **10.** $\frac{dQ}{dt} = kQ$, where k is a real number. Then Q goes to ∞ as t tend to ∞.

You can use separation of variable to obtain $Q(t) = Ce^{kt}$. The limit of $Q(t)$ depends on C, and K. For example If $C>0$, $k>0$, $\lim_{t\to\infty} Q(t) = \infty$, or if $C>0$, $k<0$, $\lim_{t\to\infty} Q(t) = 0$.

填充題 **Short answer questions** (*40* points)，每題 5 分。

(不需詳列過程，僅將答案依題號順序依序寫在答案卷第一頁上即可。)

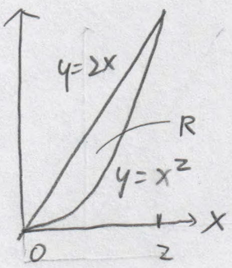
1. Let R be the plane region enclosed by the curves $y = x^2$ and $y = 2x$. Find $h_1(y)$ and $h_2(y)$ so that the double integral

$$\int\int_R xe^y \, dA = \int_0^4 \left[\int_{h_1(y)}^{h_2(y)} xe^y \, dx \right] dy.$$

Answer:___________.

Set $x^2 = 2x \Rightarrow x = 0$ or 2.

$2x - x^2 = x(2-x) \geq 0$ on $[0, 2]$, $y(0) = 0$, $y(2) = 4$



$y = 2x \Rightarrow x = \frac{1}{2}y$

$y = x^2 \Rightarrow x = \sqrt{y}$.

$\Rightarrow R = \{(x, y) \mid \frac{1}{2}y \leq x \leq \sqrt{y}, \ 0 \leq y \leq 4\}$

$\Rightarrow \boxed{h_1(y) = \frac{1}{2}y, \quad h_2(y) = \sqrt{y}}$

2. Find the general solution of the differential equation $y' = xy^2$.

Answer:___________.

Apply separation of variables to $\frac{dy}{dx} = xy^2$

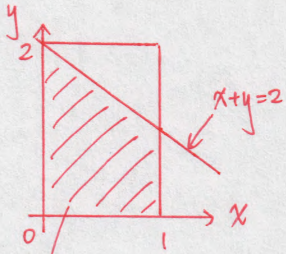
$\Rightarrow \int \frac{dy}{y^2} = \int x \, dx$

$\Rightarrow -\frac{1}{y} = \frac{1}{2}x^2 + C_0$

$\Rightarrow y = \frac{-1}{\frac{1}{2}x^2 + C_0} = \frac{-2}{x^2 + C}$

$\Rightarrow \boxed{y = \frac{-2}{x^2 + C}, \quad C \in \mathbb{R}}$

3. Let $f(x, y) = xy$ be the joint probability density function for the random variables X and Y on $D = \{(x, y)|0 \le x \le 1; 0 \le y \le 2\}$. Find the probability $P(X + Y \le 2)$. Answer:__________.



$R = \{(x,y) \mid 0 \le y \le 2-x,\ 0 \le x \le 1\}$

$$
\begin{aligned}
P(X+Y \le 2) &= \int_0^1 \left[\int_0^{2-x} xy\, dy \right] dx \\
&= \int_0^1 \left[\tfrac{1}{2} x y^2 \Big|_{y=0}^{y=2-x} \right] dx \\
&= \int_0^1 \tfrac{1}{2} x (2-x)^2\, dx \\
&= \int_0^1 \tfrac{1}{2} (4x - 4x^2 + x^3)\, dx \\
&= \left[x^2 - \tfrac{2}{3} x^3 + \tfrac{1}{8} x^4 \right]_0^1 \\
&= 1 - \tfrac{2}{3} + \tfrac{1}{8} = \tfrac{11}{24}
\end{aligned}
$$

4. Find the average value of the function $f(x, y) = ye^x$ over the plane region $\{(x, y) : 1 \le x \le 3, 0 \le y \le 1\}$. Answer:__________.

average value is defined to be

$$
\begin{aligned}
\frac{\int_1^3 \int_0^1 f(x,y)\, dy dx}{(3-1)\cdot(1-0)} &= \frac{1}{2} \int_1^3 \int_0^1 y e^x\, dy dx \\
&= \frac{1}{2} \int_1^3 e^x \tfrac{1}{2} y^2 \Big|_0^1 dx \\
&= \frac{1}{4} \int_1^3 e^x\, dx \\
&= \frac{1}{4} (e^3 - e)
\end{aligned}
$$

5. Suppose that X is a Normal random variable with mean 2 and standard deviation 2 (variance 4) and Z is a Normal random variable with mean 0 and standard deviation 1 (variance 1). Find $P\{X > 1\}$ (you may need $P\{Z < -0.5\} \approx 0.3085$). Answer:__________.

We like to normalize X into a standard normal random variable.

Therefore,

$$P[X>1] = P\left[\frac{X-2}{2} > \frac{1-2}{2}\right]$$
$$= P[Z > -0.5]$$
$$= 1 - P[Z \leq -0.5]$$
$$\approx 1 - 0.3085$$
$$= 0.6915$$

6. $y = Ce^{2x} + x^2 + x$ is a solution of $y' - 2y = 1 - 2x^2$. Find C, if $y(1) = 3$.

Answer:__________.

$$3 = y(1) = Ce^2 + 1^2 + 1 = Ce^2 + 2$$

so $C = \frac{1}{e^2}$

7. You throw a fair dice. If it appears n (n could be 1, 2, 3, 4, 5, 6), then you get a reward $10 \times n$. What is the expected value?

Answer: 35.

The expected value is

$$\frac{1}{6}\cdot 10\cdot 1+\frac{1}{6}\cdot 10\cdot 2+\frac{1}{6}\cdot 10\cdot 3+\frac{1}{6}\cdot 10\cdot 4+\frac{1}{6}\cdot 10\cdot 5+\frac{1}{6}\cdot 10\cdot 6$$

$$=\frac{1}{6}\cdot 10\cdot(1+2+3+4+5+6)$$

$$=\frac{1}{6}\cdot 10\cdot 21$$

$$=35$$

8. The population density of a certain city (number of people per square mile) is described by the function

$$f(x,y) = 10,000e^{-2x}e^{-5y}.$$

What is the population inside the rectangular area:

$$R = \{(x,y) : -5 \le x \le 5, -8 \le y \le 8\}.$$

Answer: $1000(e^{10}-e^{-10})(e^{40}-e^{-40})$.

$$\int_{-8}^{8}\int_{-5}^{5} f(x,y)\,dx\,dy = \int_{-8}^{8}\int_{-5}^{5} 10,000e^{-2x}e^{-5y}\,dx\,dy$$

$$= \int_{-8}^{8} -5,000e^{-2x}\Big|_{-5}^{5} e^{-5y}\,dy$$

$$= \int_{-8}^{8} 5000(e^{10}-e^{-10})e^{-5y}\,dy$$

$$= -1000(e^{10}-e^{-10})e^{-5y}\Big|_{-8}^{8}$$

$$= 1000(e^{10}-e^{-10})(e^{40}-e^{-40})$$

計算問答證明題 **Please show all your work** (*60* points)，每題 10 分，請依題號順序依序寫在答案卷上，可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. (*10* points) The production of a certain country is given by the function

$$f(x,y) = 20x^{\frac{3}{4}}y^{\frac{1}{4}}$$

when x units of labor and y units of capital are utilized. Find the approximate change in output if the amount expended on labor is decreased from 256 to 254 units and the amount expended on capital is increased from 16 to 18 units.

$dx = \Delta x = 254 - 256 = -2$

$dy = \Delta y = 18 - 16 = 2$

$\Rightarrow \Delta f \approx df = f_x(256,16)\,dx + f_y(256,16)\,dy$

with $dx = -2$, $dy = 2$

$f_x = 20 \cdot \frac{3}{4} x^{\frac{-1}{4}} y^{\frac{1}{4}} = 15 x^{\frac{-1}{4}} y^{\frac{1}{4}}$

$f_y = 20 \cdot x^{\frac{3}{4}} \cdot \frac{1}{4} y^{\frac{-3}{4}} = 5 x^{\frac{3}{4}} y^{\frac{-3}{4}}$

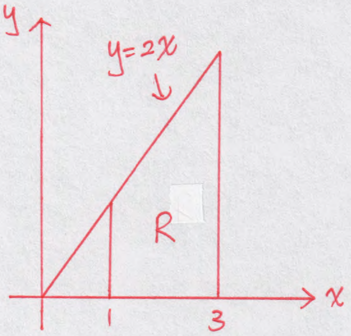
$f_x(256,16) = 15(256)^{\frac{-1}{4}}(16)^{\frac{1}{4}} = 15 \cdot \frac{1}{4} \cdot 2 = \frac{15}{2}$

$f_y(256,16) = 5(256)^{\frac{3}{4}}(16)^{\frac{-3}{4}} = 5 \cdot 64 \cdot \frac{1}{8} = 40$

$\Rightarrow \Delta f \approx \frac{15}{2} \cdot (-2) + 40 \cdot (2) = -15 + 80 = 65$

The output increases approximately by an amount of 65 units.

2. (*10* points) Find the volume of the solid bounded above by the surface $z = f(x, y)$ and below by the plane region R, where $f(x, y) = \ln x$ and R is bounded by the graphs $y = 2x$ and $y = 0$ from $x = 1$ to $x = 3$.



$R = \{ (x,y) \mid 0 \le y \le 2x, \quad 1 \le x \le 3 \}$

$$\text{volume} = \iint_R \ln x \, dA$$

$$= \int_1^3 \left[\int_{y=0}^{y=2x} \ln x \, dy \right] dx$$

$$= \int_1^3 \left[y \ln x \right]_{y=0}^{y=2x} dx$$

$$= \int_1^3 2x \ln x \, dx$$

$$= x^2 \cdot \ln x \Big|_1^3 - \int_1^3 x^2 \cdot \frac{1}{x} \, dx$$

$$= 9 \ln 3 - \left[\frac{1}{2} x^2 \right]_1^3$$

$$= 9 \ln 3 - 4$$

3. (*10* points) In a certain chemical reaction, a substance is converted into another substance at a rate proportional to the square of the amount of the first substance present at any time t. Initially ($t = 0$), 100 g of the first substance was present; 1 hr later, only 20 g of it remained. Find an expression that gives the amount of the first substance at any time t. What is the amount present after 2 hr?

Let $Q(t)$ denote the amount of the first substance present after t hours

$$\begin{cases} \frac{dQ}{dt} = -kQ^2 & k>0 \text{ constant} \\ Q(0) = 100, & Q(1) = 20 \end{cases}$$

Solve for $Q(t)$: Apply separation of variables

$$\Rightarrow \int \frac{dQ}{Q^2} = \int -k\,dt$$

$$\Rightarrow -\frac{1}{Q} = -kt + C$$

$$t=0 \Rightarrow \frac{-1}{100} = 0 + C \Rightarrow C = \frac{-1}{100}$$

$$t=1 \Rightarrow \frac{-1}{20} = -k - \frac{1}{100} \Rightarrow k = \frac{1}{20} - \frac{1}{100} = \frac{1}{25}$$

$$\Rightarrow -\frac{1}{Q} = \frac{-1}{25}t - \frac{1}{100} = \frac{-1}{100}(4t+1)$$

$$\Rightarrow Q(t) = \frac{100}{4t+1}$$

$$\Rightarrow Q(2) = \frac{100}{9}$$

$$\boxed{Q(t) = \frac{100}{4t+1}, \quad Q(2) = \frac{100}{9} \text{ (g)}}$$

4. (*10* points) Use Euler's method with $n = 4$ to obtain approximations to the solution to $y' = 16x^2 + 4y$ with $y(0) = 1$ over the interval $[0, 1]$. Please write you answer in very details.

Here $x_0 = 0$, $b = 1$, $n = 4$, so $h = \frac{1-0}{4} = \frac{1}{4}$

Therefore $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$, $x_4 = 1$

Let $F(x.y) = 16x^2 + 4y$

$x_0 = 0$, $y_0 = y(0) = 1$

$x_1 = \frac{1}{4}$, $y_1 = y_0 + hF(x_0.y_0) = 1 + \frac{1}{4}F(0,1) = 2$

$x_2 = \frac{1}{2}$, $y_2 = y_1 + hF(x_1, y_1) = 2 + \frac{1}{4}F(\frac{1}{4}, 2) = \frac{17}{4}$

$x_3 = \frac{3}{4}$, $y_3 = y_2 + hF(x_2, y_2) = \frac{17}{4} + \frac{1}{4}F(\frac{1}{2}, \frac{17}{4}) = \frac{19}{2}$

$x_4 = 1$, $y_4 = y_3 + hF(x_3.y_3) = \frac{19}{2} + \frac{1}{4}F(\frac{3}{4}, \frac{19}{2}) = \frac{85}{4}$

5. (*10* points) Solve the following first-order differential equation:

$$y' = \frac{e^{2x}}{y^4}$$

with $y(0) = 2$.

We use the separation of variable to solve this.

$$\frac{dy}{dx} = \frac{e^{2x}}{y^4}$$

$$\int y^4 \, dy = \int e^{2x} \, dx$$

$$\frac{1}{5} y^5 = \frac{1}{2} e^{2x} + C$$

By $y(0) = 2$, we obtain

$$\frac{1}{5} \cdot 32 = \frac{1}{2} + C, \quad \text{so} \quad C = \frac{59}{10}.$$

Hence $\frac{1}{5} y^5 = \frac{1}{2} e^{2x} + \frac{59}{10}$.

Therefore $y = \left(\frac{5}{2} e^{2x} + \frac{59}{2} \right)^{\frac{1}{5}}$.

6. (*10* points) Suppose that X is a normal random variable and has the following density function:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\left(\frac{-x^2}{2}\right)}.$$

Calculate $\mathrm{Var}(X)$. Hint: By the symmetry of f, we have $EX = 0$. (Please do **NOT** show this, but use it.) We also like to remind you that $\int_{-\infty}^{\infty} f(x)dx = 1$. In order to get points, you should show all your calculations and write things clearly. Showing your calculations is a **MUST**, bad explanation may cause zero point:(

By the formula of variance we have

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-0)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 2\int_0^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

We use integration by parts

$u = x$, $dv = \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$du = dx$, $v = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\mathrm{Var}(X) = 2 \lim_{b\to\infty} \int_0^b x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2 \lim_{b\to\infty} \left(-x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_0^b + \int_0^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right)$$

$$= 2 \lim_{b\to\infty} \int_0^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 1$$

(試題結束)