管院微積分聯合教學　會考六　**Jun 13, 2017**

考試時間 120 分鐘，題目卷為兩張紙，共三頁，滿分 120 分。所有題目的答案都請依題號順序依序寫在答案卷上，而是非與填充題必須寫在第一頁。答案卷務必寫學號、姓名，題目卷不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘內不得離場。考試期間禁止使用字典、計算機、任何通訊器材並請勿自行攜帶任何紙張，違者成績以零分計算，監試人員不得回答任何關於試題的疑問。**Questions are to be answered on the answer sheet provided.**

是非題 **True or False** (*20* points)，請答 **T** (True) 或 **F** (False)。每題 2 分。
(不需詳列過程，請依題號順序依序寫在答案卷第一頁上。)

F **1.** If $0 < r < 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{r^n} = \frac{1/r}{1-1/r} = \frac{1}{r-1}$.

公比 $\frac{1}{r} > 1$，$\sum_{n=1}^{\infty} \frac{1}{r^n}$ 發散.

T **2.** The series $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n^2}$ converges for all x.

$\sum_{n=1}^{\infty} \frac{x}{n^2} = x \sum_{n=1}^{\infty} \frac{1}{n^2}$，其中 $\sum_{n=1}^{\infty} \frac{1}{n^2}$ 為 $p=2>1$ 的 p-級數，收斂

故對所有的 x，$\sum_{n=1}^{\infty} \frac{x}{n^2}$ 收斂.

T **3.** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\sum b_n$ is convergent and $b_n \geq a_n$ for all n, then $\sum a_n$ is convergent.

因為均為正項且 $b_n > a_n$，當 Σb_n 收斂可導出 Σa_n 亦收斂.

T **4.** Using Newton's method with $f(x) = x^3 - 10$ and initial guess $x_0 = 2$, the first iterative estimate x_1 of $\sqrt[3]{10}$ is $\dfrac{13}{6}$.

因為 $\sqrt[3]{10}$ 是 $f(x) = x^3 - 10$ 的根，$\sqrt[3]{10} \approx x_1 = x_0 - \left.\dfrac{f(x_0)}{f'(x_0)}\right|_{x_0=2} = 2 - \dfrac{2^3-10}{3(2)^2}$

$= 2 - \dfrac{-2}{12} = 2 + \dfrac{1}{6} = \dfrac{13}{6}$

F **5.** The graph of $f(x) = x - \tan x$ is increasing on $\left(0, \dfrac{\pi}{2}\right)$.

$f'(x) = 1 - \sec^2 x < 0$, $0 < x < \frac{\pi}{2}$

得 f 在 $(0, \frac{\pi}{2})$ 上遞減.

T **6.** If $\sum a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$.

See the theorem on page 725.

F **7.** $\sum_{n=1}^{\infty} \frac{5n^2+1}{8n^2-1}$ converges.

$$\lim_{n\to\infty} \frac{5n^2+1}{8n^2-1} = \lim_{n\to\infty} \frac{5+\frac{1}{n^2}}{8-\frac{1}{n^2}} = \frac{5}{8} \neq 0$$

so by the test for divergence, the series diverges.

T **8.** If $f(x,y) = x^2y + e^{x^2+y^2}$, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 + 2y e^{x^2+y^2}\right) = 2x + 4xy e^{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy + 2x e^{x^2+y^2}\right) = 2x + 4xy e^{x^2+y^2}$$

F **9.** If f is a probability density function on $[a,b]$, then $0 \le f(x) \le 1$ on $[a,b]$.

f is a probability density function on $[a,b]$

if $f(x) \ge 0$ on $[a,b]$ and $\int_a^b f(x)dx = 1$.

Then condition $0 \le f(x) \le 1$ does not true for a general.

T **10.** The series $\sum_{n=0}^{\infty} n!(x-1)^n$ converges only at $x=1$.

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0$$

填充題 **Short answer questions** (*40* points)，每題 5 分。

(不需詳列過程，僅將答案依題號順序依序寫在答案卷第一頁上即可。)

1. Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n (x+2)^n}{3^{n+1}}$.

Answer：___________.

$$R = \lim_{n\to\infty} \left| \frac{(-1)^n}{3^{n+1}} \cdot \frac{3^{n+2}}{(-1)^{n+1}} \right| = \lim_{n\to\infty} 3 = 3, \text{ 得}$$

收斂區間 $(-2-3, -2+3) = (-5, 1)$

2. Find the Taylor series of the function $f(x) = \frac{1}{1+x}$ at $x = 1$.

Answer：___________.

$$\frac{1}{1+x} = \frac{1}{2+(x-1)} = \frac{1}{2[1-(-\frac{x-1}{2})]} = \frac{1}{2}\,\frac{1}{1-(-\frac{x-1}{2})}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x-1}{2}\right)^n, \quad -1 < -\frac{x-1}{2} < 1$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-1)^n, \quad -1 < x < 3$$

3. Evaluate the definite integral $\displaystyle\int_0^{\pi/2} (\sin x + \cos x)^2\, dx$.

Answer : ____________.

$$\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 dx = \int_0^{\frac{\pi}{2}} (\sin^2 x + 2\sin x \cos x + \cos^2 x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (1 + 2\sin x \cos x)\, dx$$

$$\underset{\substack{u=\sin x \\ du=\cos x\, dx}}{=} \quad x + (\sin x)^2 \Big|_0^{\pi/2} = \frac{\pi}{2} + 1$$

4. Evaluate $\displaystyle\lim_{x\to 0^-} \frac{e^{x^2} + x - 1}{1 - \sqrt{1-x^2}}$.

Answer : ____________.

By l'Hôpital's rule,

$$\lim_{x\to 0^-} \frac{e^{x^2} + x - 1}{1 - \sqrt{1-x^2}} = \lim_{x\to 0^-} \frac{2xe^{x^2} + 1}{\frac{x}{\sqrt{1-x^2}}} = \frac{1}{0^-} = -\infty$$

5. Find the equation of the tangent line to the graph of the function $f(x) = \tan 2x$ at the point $\left(\frac{\pi}{8}, 1\right)$.

Answer : ____________.

The slope of the tangent line $f'(x) = 2\sec^2 2x$

at the point $(\frac{\pi}{8}, 1)$ is given by $f'(\frac{\pi}{8}) = 2\sec^2\frac{\pi}{4} = 2(\sqrt{2})^2 = 4$

Therefore, the required equation is give by

$$y - 1 = 4(x - \tfrac{\pi}{8}) \quad (\text{or } y = 4x + (1 - \tfrac{\pi}{2})).$$

6. Let $h(x) = e^{\sin 3x + \cos 3x}$. Find $h'(x)$.

Answer : ____________.

Using the Chain Rule, we find

$$h'(x) = e^{\sin 3x + \cos 3x} \cdot \frac{d}{dx}(\sin 3x + \cos 3x)$$

$$= e^{\sin 3x + \cos 3x}\left[(\cos 3x) \cdot \frac{d}{dx}(3x) - (\sin 3x) \cdot \frac{d}{dx}(3x)\right]$$

$$= 3(\cos 3x - \sin 3x)\, e^{\sin 3x + \cos 3x}.$$

7. Find the sum of the following series if it exists.

$$\sum_{n=1}^{\infty}\left[\frac{1}{2^n}-\frac{1}{n(n+1)}\right]$$

Answer : ___________.

$$\sum_{n=1}^{\infty}\left[\frac{1}{2^n}-\frac{1}{n(n+1)}\right]=\sum_{n=1}^{\infty}\frac{1}{2^n}-\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right)$$

$$=\frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right)-\sum_{n=1}^{\infty}\frac{1}{n}+\sum_{n=1}^{\infty}\frac{1}{n+1}$$

$$=1-\left(1+\cancel{\frac{1}{2}}+\cancel{\frac{1}{3}}+\cdots\right)+\left(\cancel{\frac{1}{2}}+\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\cdots\right)$$

$$=1-1=0$$

8. Suppose f'' is continuous on $[1,3]$ and $f(1)=3$, $f(3)=4$, $f'(1)=1$ and $f'(3)=2$. Evaluate $\displaystyle\int_1^3 xf''(x)\,dx$.

Answer : ___________.

$$\int_1^3 xf''(x)\,dx=\left[xf'(x)\right]_1^3-\int_1^3 f'(x)\,dx$$

$$=(3\cdot 2-1\cdot 1)-\left[f(x)\right]_1^3$$

$$=5-(4-3)=4$$

計算問答證明題 **Please show all your work** (*60* points)，每題 10 分，請依題號順序依序寫在答案卷上，可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. (*10* points) Use a sixth-degree Taylor polynomial to approximate $\displaystyle\int_0^{0.1} \frac{1}{\sqrt{1+x^3}}\,dx$. Express your answer as a sum of numbers.

令 $f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$, 得 $f'(x) = -\frac{1}{2}(1+x)^{-\frac{3}{2}}$, $f''(x) = \frac{3}{4}(1+x)^{-\frac{5}{2}}$ 且

$f(0) = 1$, $f'(0) = -\frac{1}{2}$, $f''(0) = \frac{3}{4}$.

故 $f(x) \approx P_2(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2$.

代 x^3, 得 $\frac{1}{\sqrt{1+x^3}} = f(x^3) \approx P_2(x^3) = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6$, 6階泰勒多項式.

因此 $\int_0^{0.1} \frac{1}{\sqrt{1+x^3}}\,dx \approx \int_0^{0.1} (1 - \frac{1}{2}x^3 + \frac{3}{8}x^6)\,dx = x - \frac{1}{8}x^4 + \frac{3}{56}x^7 \Big|_0^{0.1}$

$= 0.1 - \frac{1}{8}(0.1)^4 + \frac{3}{56}(0.1)^7$.

2. (*10* points) The revenue of McMenamy's Fish Shanty, located at a popular summer resort, is approximately $R(t) = 2\left(5 - 4\cos\frac{\pi}{6}t\right)$, $0 \le t \le 12$, during the t-week ($t = 1$ corresponds to the first week of June), where R is measured in thousands of dollars. When is the weekly revenue increasing most rapidly?

依題意,需求 $R'(t) = 2(4)(\frac{\pi}{6})\sin(\frac{\pi}{6}t) = \frac{4\pi}{3}\sin(\frac{\pi}{6}t)$, $0 \le t \le 12$, 的最大值.

令 $R''(t) = \frac{4\pi}{3}\frac{\pi}{6}\cos(\frac{\pi}{6}t) = \frac{2\pi^2}{9}\cos(\frac{\pi}{6}t) = 0$, 得 $\frac{\pi}{6}t = \frac{\pi}{2}$, $\frac{\pi}{6}t = \frac{3\pi}{2}$.

即 $R'(t)$ 的臨界點 $t=3$ 和 $t=9$,

又 $R'(t)$ 在 $[0,12]$ 上連續,故 $R'(t)$ 的最大值發生在端点或臨界点,比較 $R'(0)=0$, $R'(3)=\frac{4\pi}{3}$, $R'(9)=-\frac{4\pi}{3}$, $R'(12)=0$,

得 $t=3$ 時有最大值.

3. (*10* points) The weekly closing price of HAL Corporation stock in week t is approximated by the rule $f(x) = 30 + t\sin\frac{\pi}{6}t$, $0 \le t \le 15$, where $f(t)$ is the price (in dollars) per share. Find the average weekly closing price of the stock over the 15-week period.

平均週收盤價 $= \frac{1}{15}\int_0^{15}(30 + t\sin\frac{\pi}{6}t)\,dt$

$$= \frac{1}{15}\int_0^{15} 30\,dt + \frac{1}{15}\int_0^{15} t\sin\frac{\pi}{6}t\,dt$$

$$\begin{pmatrix} u=t, & dv=\sin\frac{\pi}{6}t \\ du=dt, & v=-\frac{6}{\pi}\cos\frac{\pi}{6}t \end{pmatrix} = \frac{1}{15}(30t)\Big|_0^{15} + \frac{1}{15}\left[-\frac{6}{\pi}t\cos\frac{\pi}{6}t\Big|_0^{15} + \frac{6}{\pi}\int_0^{15}\cos\frac{\pi}{6}t\,dt\right]$$

$$= 30 + \frac{1}{15}\left[-\frac{6}{\pi}(15)\cos\left(\frac{5\pi}{2}\right) + \left(\frac{6}{\pi}\right)^2\sin\left(\frac{5\pi}{2}\right)\right]$$

$$= 30 + \frac{1}{15}\left[-\frac{90}{\pi}(0) + \frac{36}{\pi^2}(1)\right] = 30 + \frac{12}{5\pi^2}$$

4. (*10* points) It takes x units of labor and y units of capital to produce

$$f(x, y) = 100x^{3/4}y^{1/4}.$$

If a unit of labor costs \$200 and a unit of capital costs \$100 and \$200,000 is budgeted for production, determine how many units should be expended on labor and how many units should be expended on capital in order to maximize production (use the method of Lagrange multipliers ONLY).

Since the budget is 200,000

$$200x + 100y = 200{,}000$$

namely $2x + y = 2{,}000$

we like to maximize $f(x,y) = 100\, x^{\frac{3}{4}} y^{\frac{1}{4}}$

subject to $g(x,y) = 2x + y - 2{,}000 = 0$,

Let $F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$

$F_x(x, y, \lambda) = 75\, x^{\frac{1}{4}} y^{\frac{1}{4}} + 2\lambda = 0$ ①

$F_y(x, y, \lambda) = 25\, x^{\frac{3}{4}} y^{-\frac{3}{4}} + \lambda = 0$ ②

$F_\lambda(x, y, \lambda) = 2x + y - 2{,}000 = 0$ ③

by ① = ②$\cdot 2$ we have

$$75 \cdot \left(\frac{y}{x}\right)^{\frac{1}{4}} + 2\lambda = 50 \left(\frac{x}{y}\right)^{\frac{3}{4}} + 2\lambda$$

$\frac{y}{x} = \frac{50}{75} = \frac{2}{3}$, so $y = \frac{2}{3}x$

Together with ③, we obtain

$2x + \frac{2}{3}x = 2{,}000$, so $x = 750$. Hence $y = 500$.

Since $f(0, 2{,}000) = f(1{,}000, 0) = 0$,

750 units of labor and 500 units of capital maximize production.

5. (*10* points) Find the Taylor series of the function at the indicated number.

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a. $f(x) = xe^{-x}$; $x = 0$.

b. $f(x) = \ln(1+x)$; $x = 0$.

a. First, we replace x with $-x$ in the expression

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

to obtain

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots + \frac{(-x)^n}{n!} + \cdots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^n}{n!} + \cdots$$

Then multiplying both sides

$$f(x) = xe^{-x} = x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \cdots + \frac{(-1)^n x^{n+1}}{n!} + \cdots$$

b. we replace x with $1+x$ in the expression

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n + \cdots$$

to obtain

$$f(x) = \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n+1}}{n}x^n + \cdots$$

6. (*10* points) Use the integral test to determine whether

✓

$$\sum_{n=2}^{\infty} \frac{\ln n}{n}$$

converges or diverges.

we consider the function $f(x) = \frac{\ln x}{x}$,

observe that f is continuous and positive on $[2, \infty)$,

compute

$$f'(x) = \frac{x(\frac{1}{x}) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Note that $f'(x) < 0$ if $\ln x > 1$, that is $x > e$.

This shows that f is decreasing on $[3, \infty)$

Therefore, we may use the integral test,

$$\int_3^\infty \frac{\ln x}{x}\,dx = \lim_{b \to \infty} \int_3^b \frac{\ln x}{x}\,dx$$

$$= \lim_{b \to \infty} \left[\frac{1}{2} (\ln x)^2 \Big|_3^b \right]$$

$$= \lim_{b \to \infty} \frac{1}{2} \left[(\ln b)^2 - (\ln 3)^2 \right] = \infty$$

So we conclude that $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges.

(試題結束)