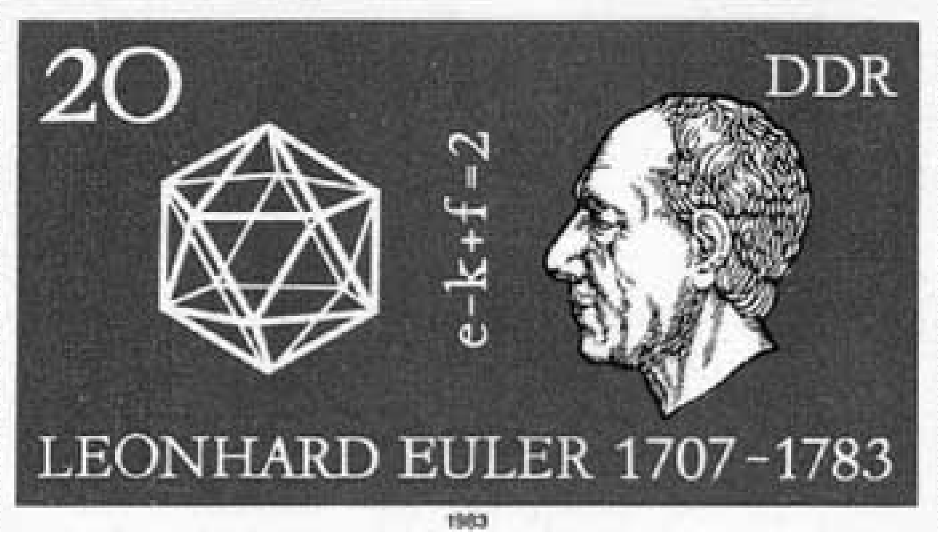
淺談尤拉公式及其應用

Euler Formula ： V-E+F=2

陳建隆(中央大學數學系)



Euler（1707～1783）瑞士數學家

圖形出自 http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Euler.html

Euler Formula

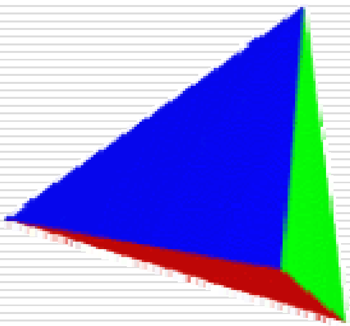
- For any convex polyhedron, the number of vertices and faces together is exactly two more than the number of edges.

$$V-E+F=2.$$

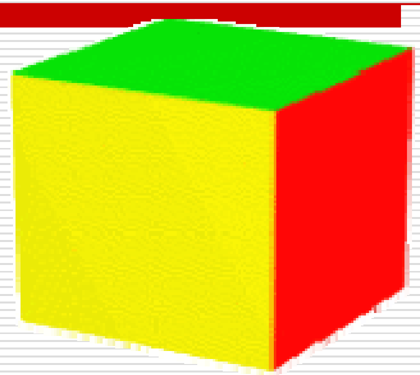
(點 – 邊 + 面 = 2)

- 正四面體: V=4(四個頂點), E=6(六個邊),
  F=4(四個面)
  4-6+4=2.

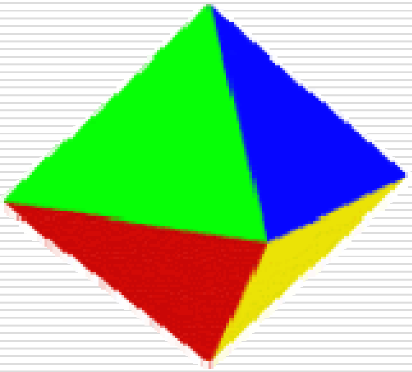
正凸多面體



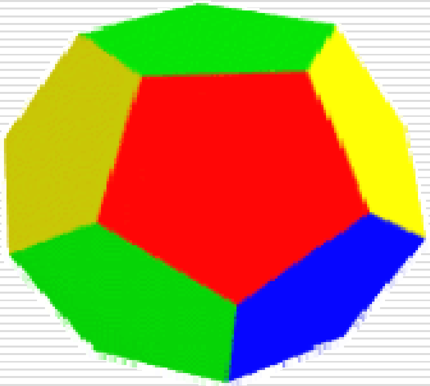
{3,3}
正四面體



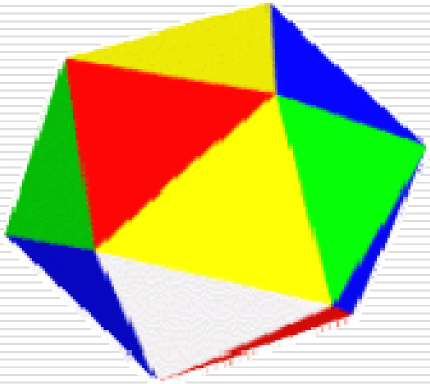
{4,3}
正六面體



{3,4}
正八面體



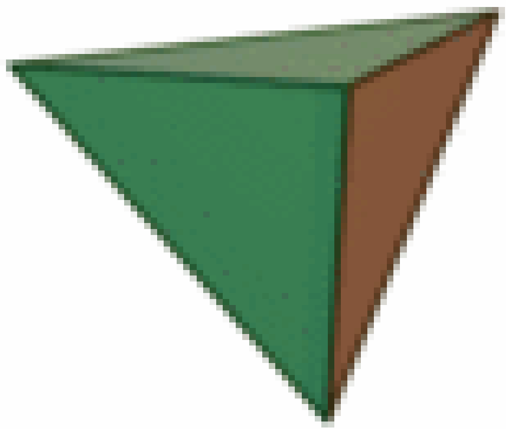
{5,3}
正十二面體



{3,5}
正二十面體

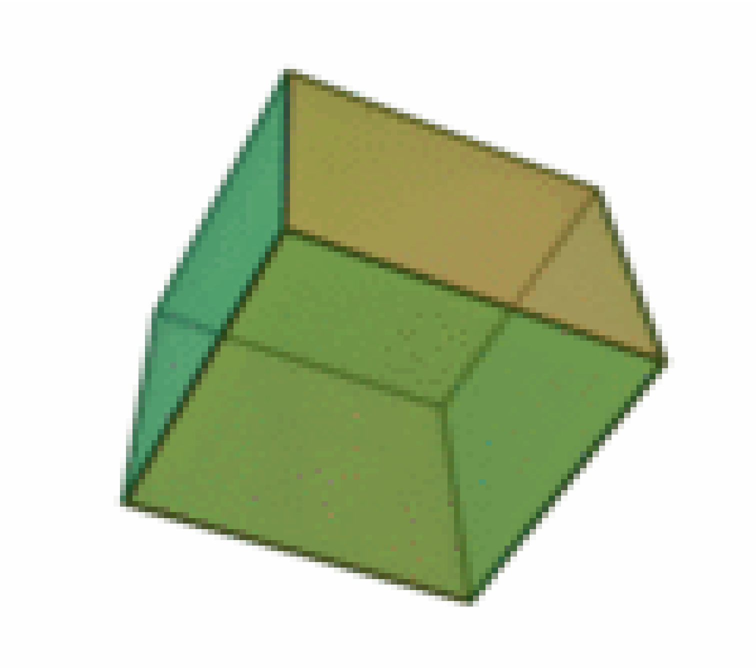
(1) 正四面體

- Tetrahedron
- V = 4
- E = 6
- F = 4
- 4 - 6 + 4 = 2



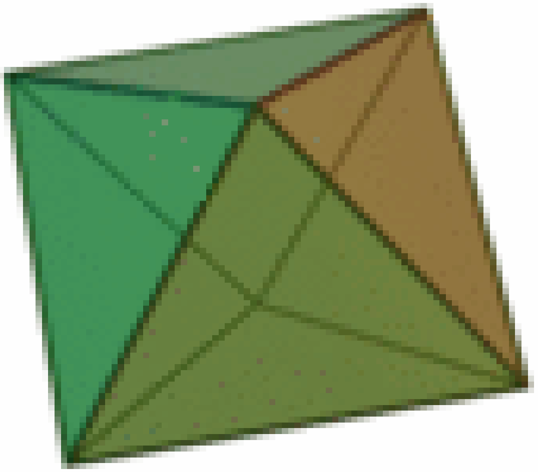
(2) 正六面體

- Cube
- V = 8
- E = 12
- F = 6
- 8 - 12 + 6 = 2



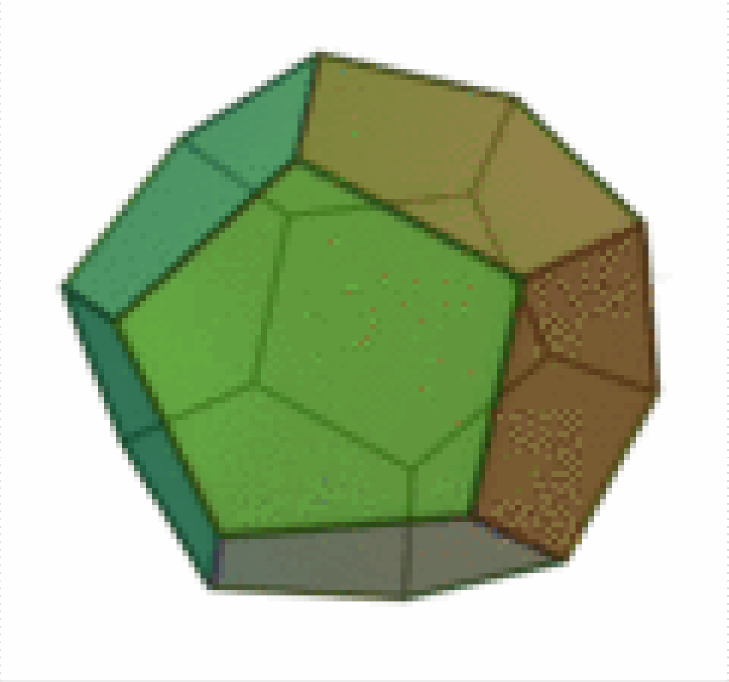
(3) 正八面體

- Octahedron
- V = 3*8/4 = 6
- E = 3*8/2 = 12
- F = 8
- 6 - 12 + 8 = 2



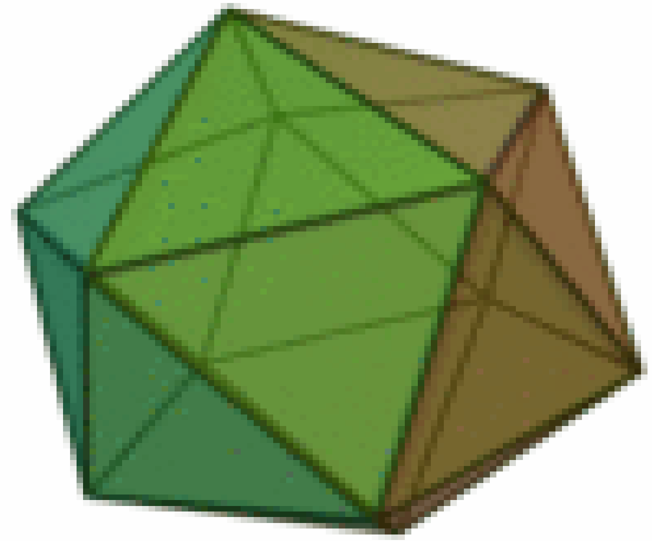
(4) 正十二面體

- Dodecahedron
- $V = 5*12/3=20$
- $E = 5*12/2=30$
- $F = 12$
- $20 - 30 + 12 = 2$



(5) 正二十面體

- Icosahedron
- V = 3*20/5=12
- E = 3*20/2=30
- F = 20
- 12 - 30 + 20 = 2



歐幾里得：凸正多面體只有五個

- Proof:

(1) 設 每 一 個頂點恰有m 個正n邊形相鄰，

則 (i) $m*(n-2)*180/n < 360$

(ii) $nF=mV=2E$

By (i) ➔ $0<(n-2)(m-2)<4$

By (ii) ➔ $F=2E/m$, $V=2E/n$

By Euler Formula, we have

$$V=4n/(2m+2n-mn)$$

$$E=2mn/(2m+2n-mn)$$

$$F=4m/(2m+2n-mn)$$

(2) (n,m) 最多可能是： $0<(n-2)(m-2)<4$

$F=4m/(2m+2n-mn)$

(3,3) ➔ $F=4*3/(2*3+2*3-3*3)=4$ 正四面體;

(3,4) ➔ $F=4*4/(2*4+2*3-4*3)=8$ 正八面體

(3,5) ➔ $F=20$ 正二十面體;

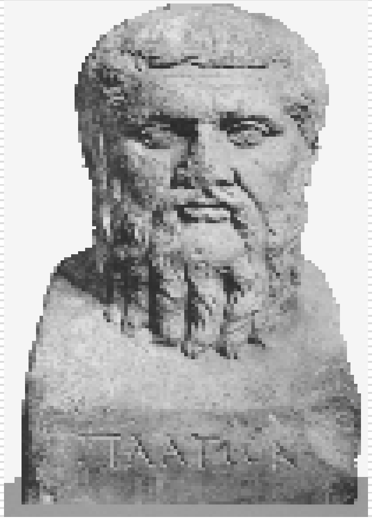
(4,3) ➔ $F=6$ 正六面體;

(5,3) ➔ $F=4*3/(2*3+2*5-3*5)=12$ 正十二面體.

柏拉圖立體

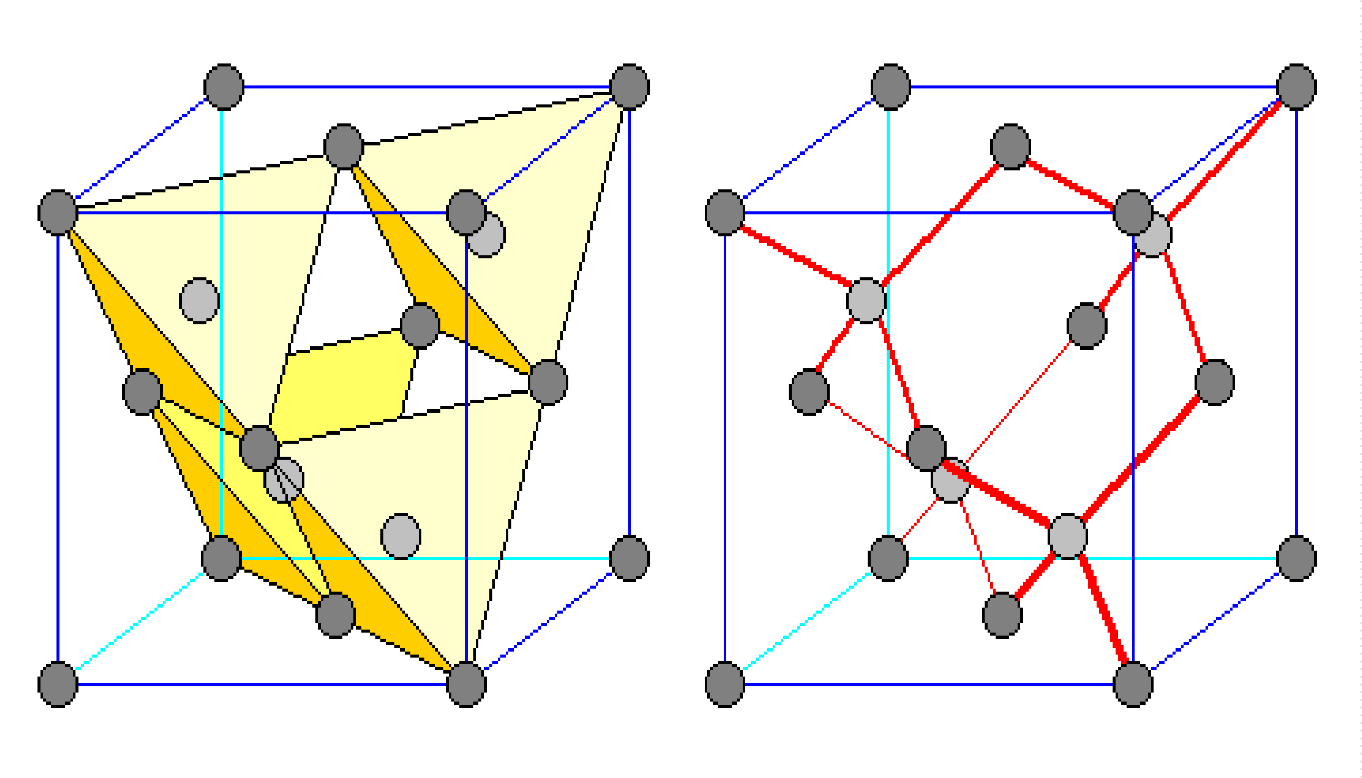
- 古希臘的哲學家中，有人認爲火、土、氣、水是自然界的四種基本元素，柏拉圖分別用正四面體、正六面體、正八面體和正二十面體來代表這四種元素，而整個宇宙的形狀就是正十二面體 。

正多面體	象徵物	正多邊形（p）	每個頂點所接的正多邊形個數（q）
正四面體	火	正三邊形	3
正六面體	土	正四邊形	3
正八面體	空氣	正三邊形	4
正十二面體	宇宙	正五邊形	3
正二十面體	水	正三邊形	5



自然物體的結構

- (1) 鑽石結構

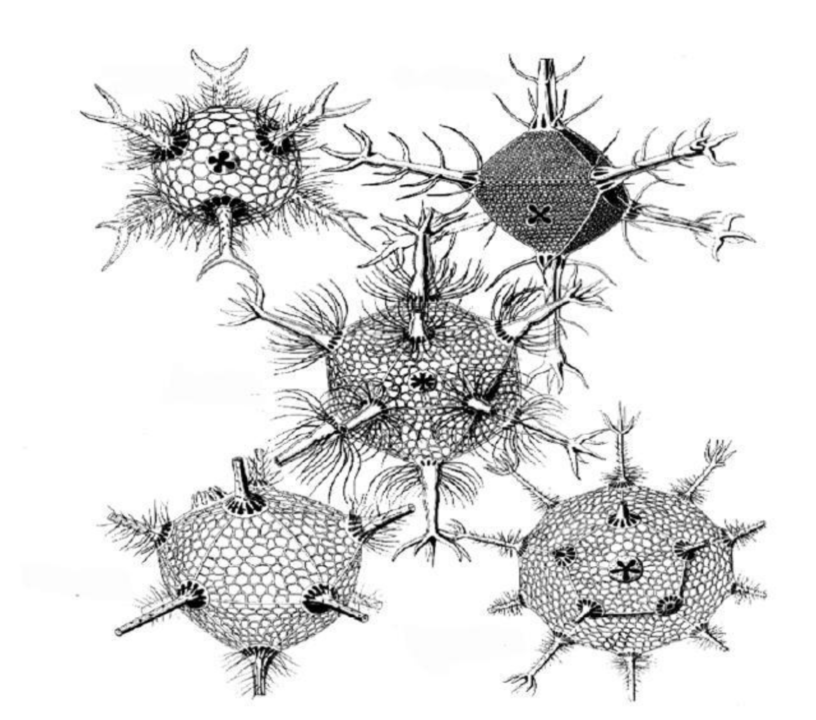


硼

- 存在：硼於地殼中之含量僅0.001% ，主要以硼酸鹽存於礦石中，由以硼砂($Na_2B_4O_7 \cdot 10H_2O$)最重要.
- 元素硼，十二個硼原子剛好形成一個正二十面體.

放射蟲的骨架

- 很多放射蟲的骨架，就是正八面、十二面和二十面體 . (圖形出自 http://www.blem.ac.cn/admin/download/khkuo.pdf)



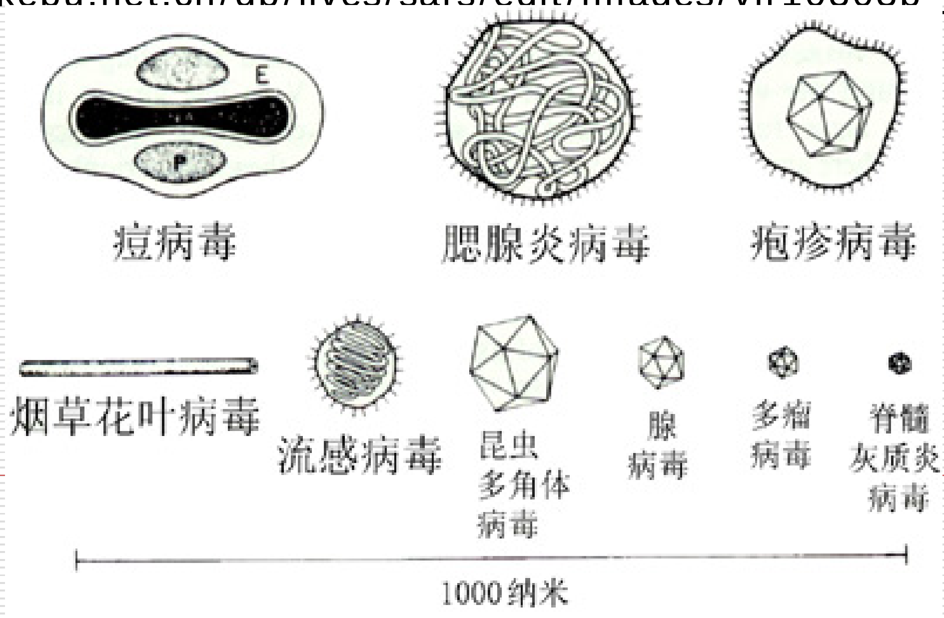
- ➔ 八面體
- ➔ 三角二十面體
- ➔ 五角十二面體

腺病毒的衣壳

- **腺病毒**的衣壳是典型的二十面體對稱:

 由252個衣壳组成，没有包膜, 腺病毒的核心是由線狀雙鏈DNA構成的, 其基因组的大小都约爲36500個核苷酸對。

(出自http://www.kepu.net.cn/gb/lives/sars/edit/images/vir10303b_pic.jpg)

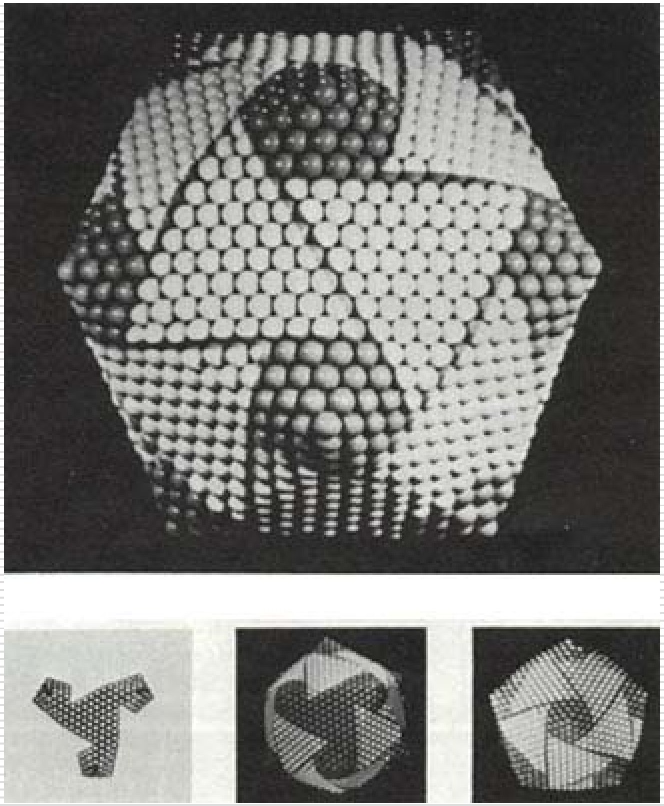


Problems

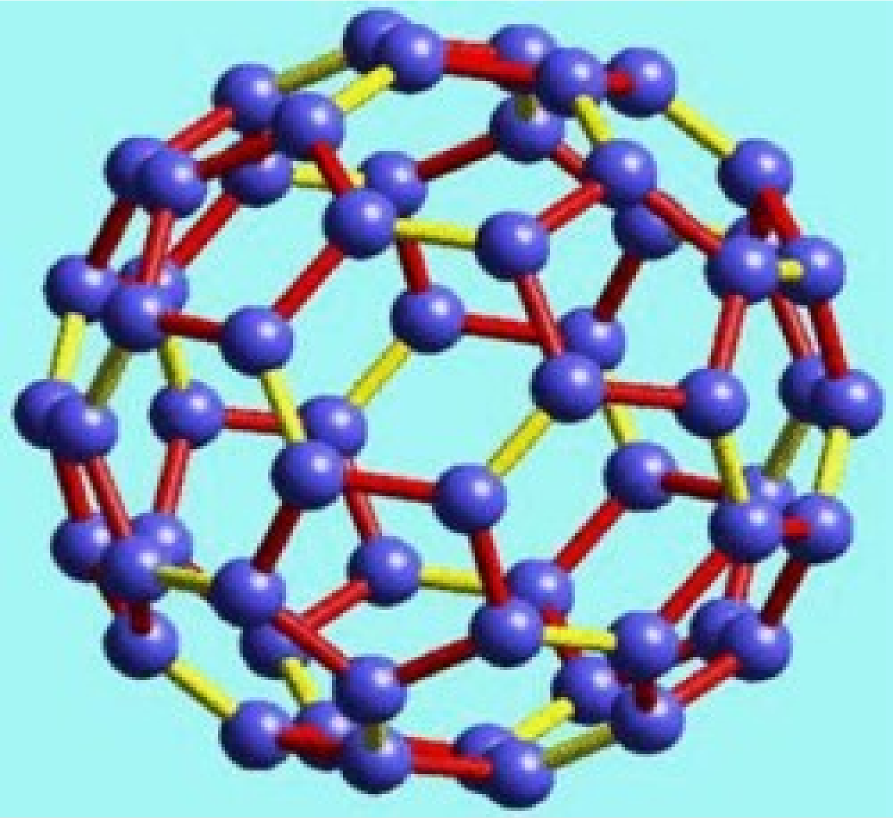
- (A) 是否存在由兩類正多邊形所組成的一凸多面對稱體？
- (B) 有否一實際自然物質其分子結構是上述凸多面對稱體？

腺病毒

- (圖形出自:http://www.kepu.net.cn/gb/lives/sars/edit/200305200048.html)



C60－碳六十 (Buckminsterfullerene)



(圖形出自 http://nano.nchc.org.tw/dictionary/c60.html)

巴克球(Bucky ball)

1967年加拿大蒙特婁世界博覽會上
的美國館，建築物高60公尺
(圖形出自 http://nano.nchc.org.tw/dictionary/c60.html)

柱頭建築

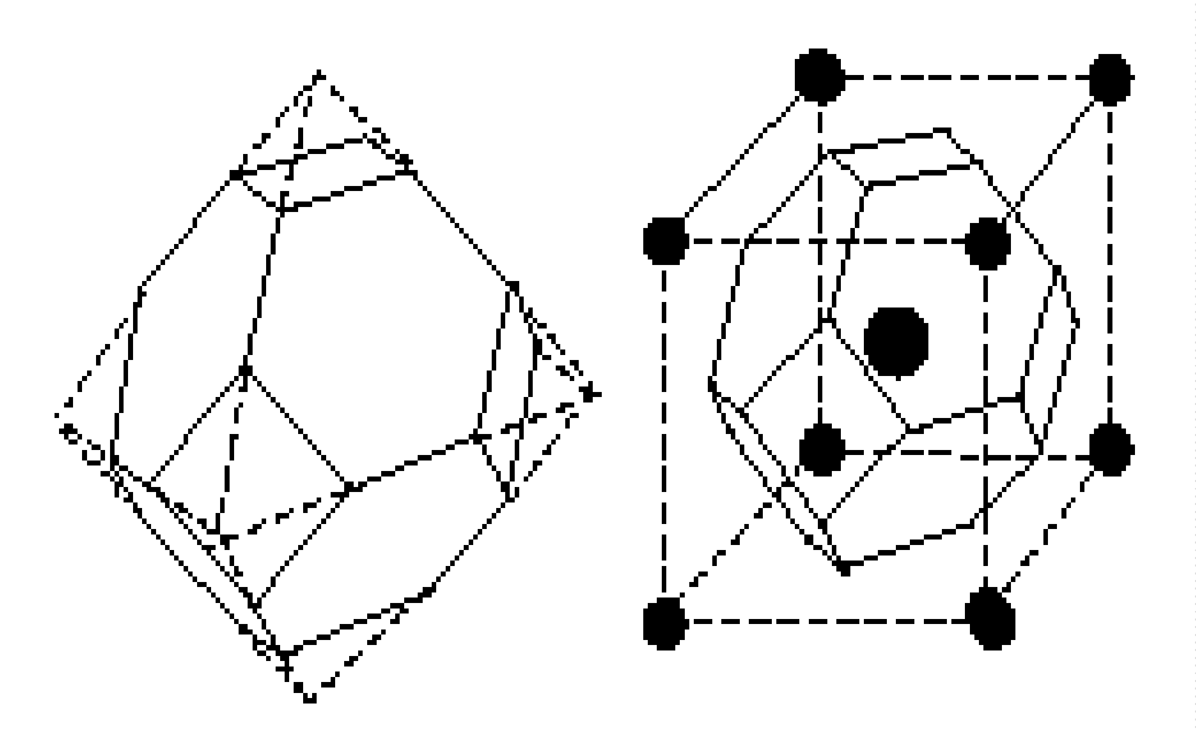
(圖形出自http://www.cabtc.gov.tw/LinFamily/4-1-3.htm)

十四面體柱頭

南瓜形柱頭

四面體柱頭

正八面體 ➔ 十四面體



十二面體

（圖形出自http://museum.pku.edu.cn/exp/mineral/mineral&crytal/6symmetry.asp）

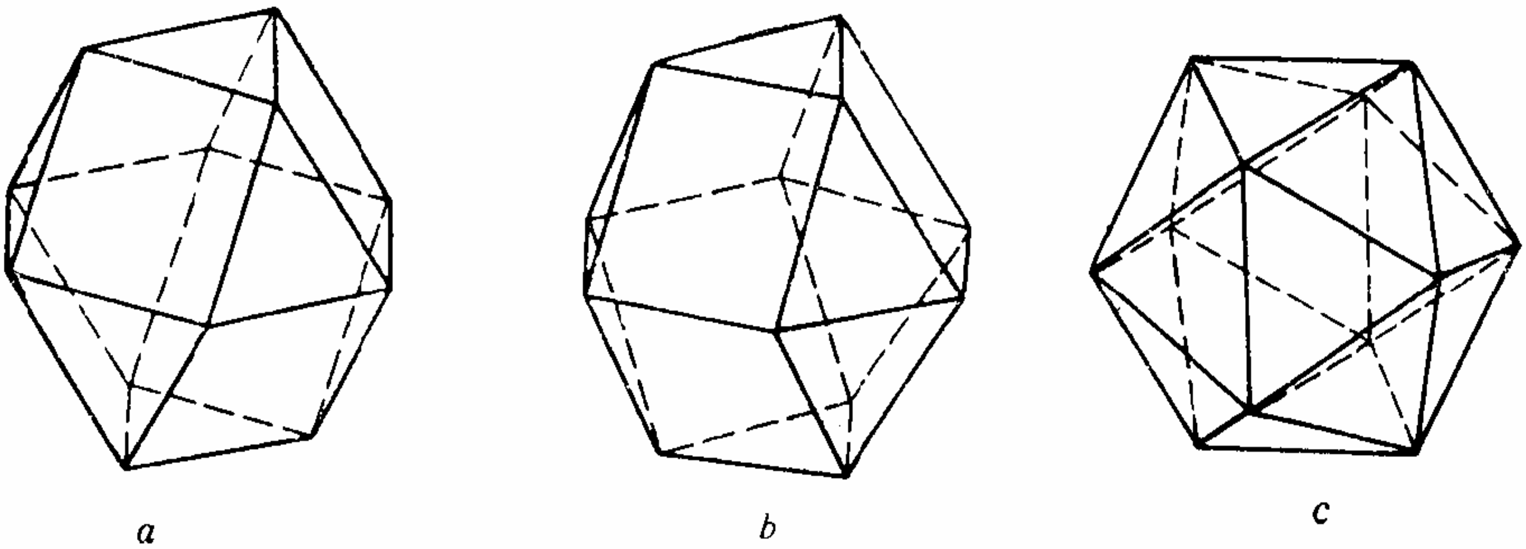
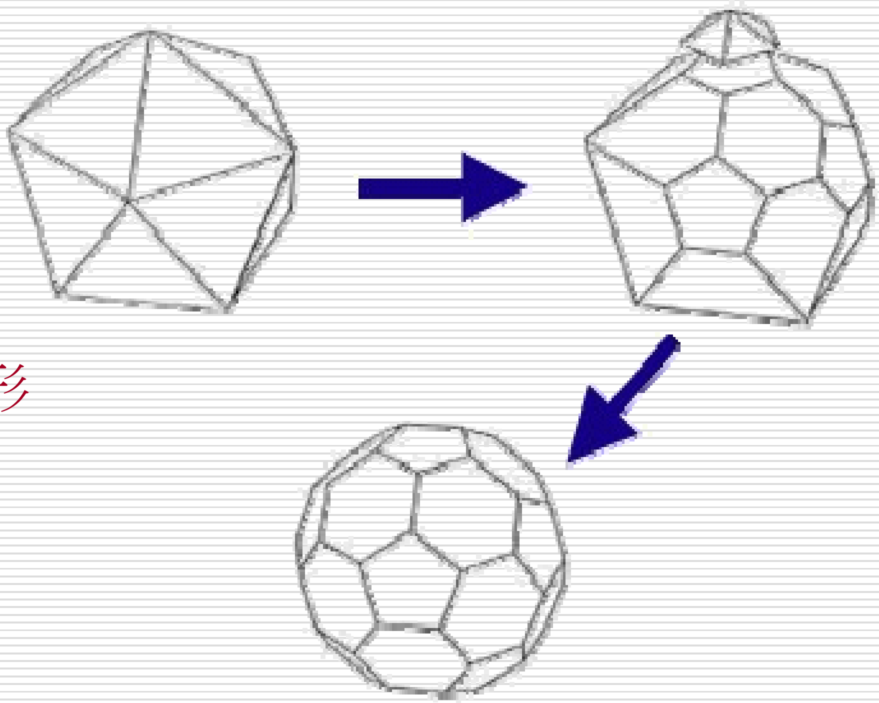


图 I—4—24 几种配位数为 12 的配位多面体

正二十面體 ➔ 三十二面體

(圖形出自http://nano.nchc.org.tw/dictionary/c60.html)



12個頂點切出12個正五邊形

20面正三角形變成20面正六邊形

對稱多面體

- 切頂正四面體 ➔八面對稱體：正六邊形4＋正三邊形4
- 切頂正六面體 ➔十四面對稱體：正六邊形6＋正三邊形 8
- 切頂正八面體 ➔十四面對稱體：正六邊形8＋正四邊形6
- 切頂二十面體 ➔ 三十二面對稱體 C60 ：
  正六邊形20＋正五邊形12 (足球結構)

C60, C70, C100, C300

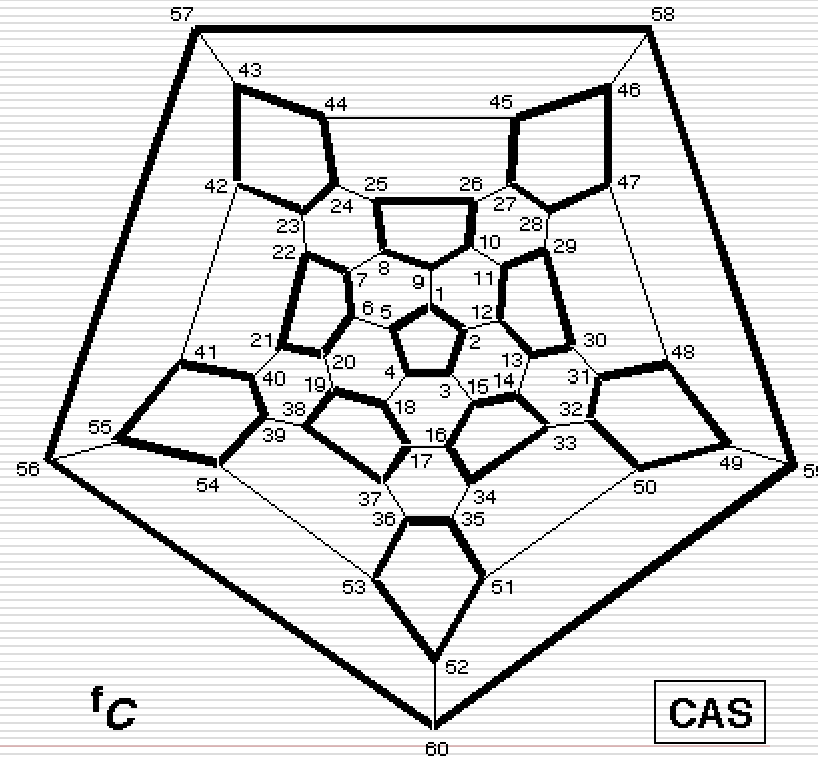
- 美國夏威夷大學的科學家早已發現四十六億年前的隕石內存在著C60、C70以及C100~C300等純碳分子，認爲是宇宙早期即已存在的物質；然而，在地球上發現C60卻是天文物理研究的收穫.

- 1985年，英國化學家柯洛托(Sir Harold W. Kroto)探索在可見光與紫外光之間，是否存在屬於微小石墨碳粒的星際塵埃光譜，在柯爾(Robert F. Curl)與史莫利(Richard E. Smalley)的協助下，以聚焦雷射蒸發石墨，再與鈍氣混合由噴嘴噴出冷卻，並以質譜儀記錄產物，測出含有偶數個碳原子的碳簇(carbon cluster);這些成就使柯洛托、柯爾及史莫利三人於1996年共同榮膺諾貝爾化學獎。

C60

(圖形出自 http://www.diederich.chem.ethz.ch/chirafull/c60cas)

- 令正五邊形有 F1 個, 正六邊形有 F2 個
  則
  (1) By Euler Formula
  V-E+F=2 and
  V=60
  3V=5F1+6F2=2E,
  V=5*F1



- (2) F1=12, F2=20

C70

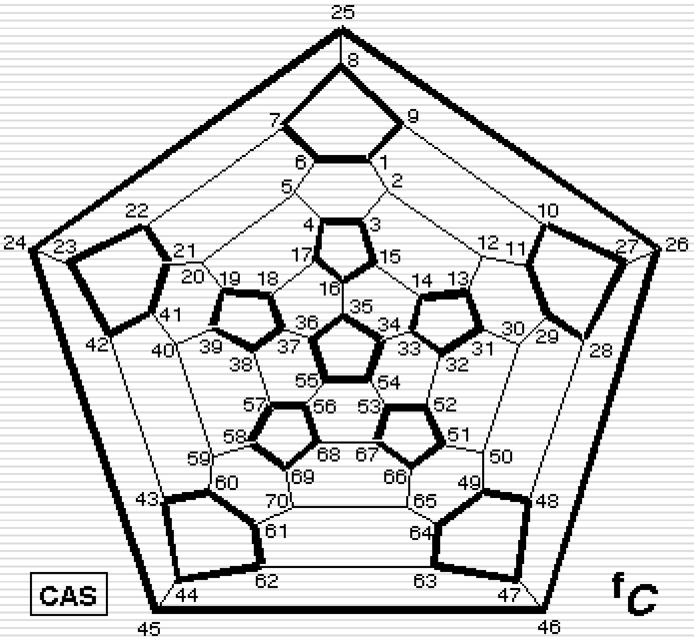
- 令正五邊形有 F1 個, 正六邊形有 F2 個 則

  (1) By Euler Formula and

  $V=70$

  $3V=5F1+6F2=2E,$

  $V=5*F1+(F2-10)*2/3$



- (2) $F1=12,\ F2=25$

(圖形出自 http://www.diederich.chem.ethz.ch/chirafull/c60cas)

C100

- 令正五邊形有 F1 個, 正六邊形有 F2 個則

  (1) V＝100

  3V＝5F1＋6F2＝2E,

  ➔ E＝150

  By Euler Formula：V-E＋F＝2

  ➔ F=F1＋F2＝52

- (2) F1＝12, F2＝40

C300

- 令正五邊形有 F1 個, 正六邊形有 F2 個則

  (1) $V=300$

  $3V=5F1+6F2=2E,$

  ➔ $E=450$

  By Euler Formula : $V-E+F=2$

  ➔ $F=F1+F2=152$

- (2) $F1=12,\ F2=140$

Proof of Euler Formula:

- (2) This argument is the planar dual to the proof by induction on faces.
- (3) If G has only one vertex, each edge is a Jordan curve, so there are E+1 faces and F+V-E=(E+1)+1-E=2. Otherwise, choose an edge *e* connecting two different vertices of G, and contract it. This decreases both the number of vertices and edges by one, and the result then holds by induction.

Problem

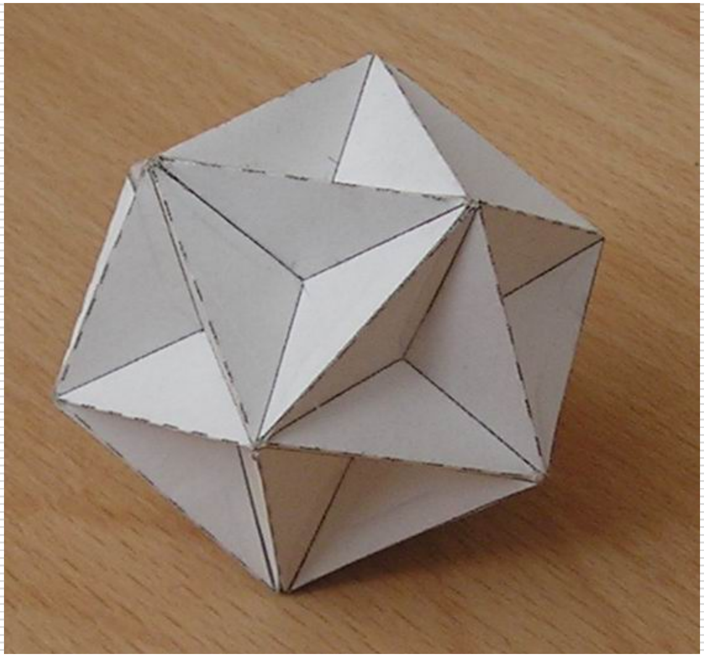
- 空間中非凸多面體是否滿足公式：

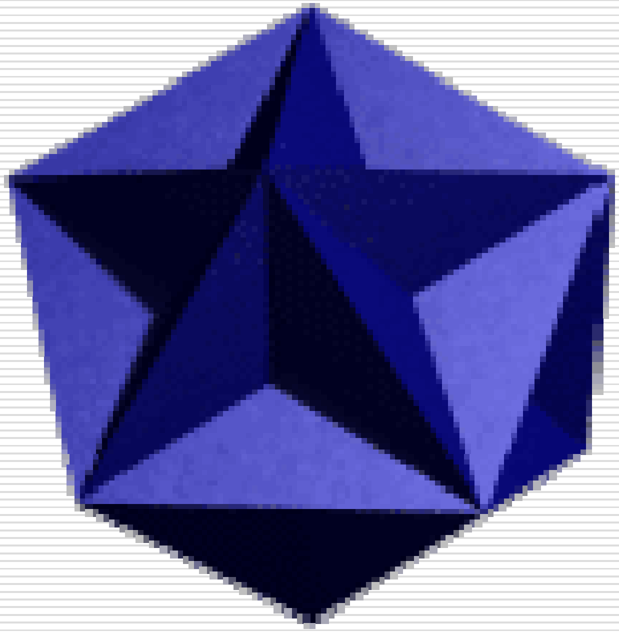
  V-E+F=2？

Dodecahedron

- There exist polytopes which do not satisfy the polyhedral formula, the most prominent of which are the great dodecahedron and small stellated dodecahedron

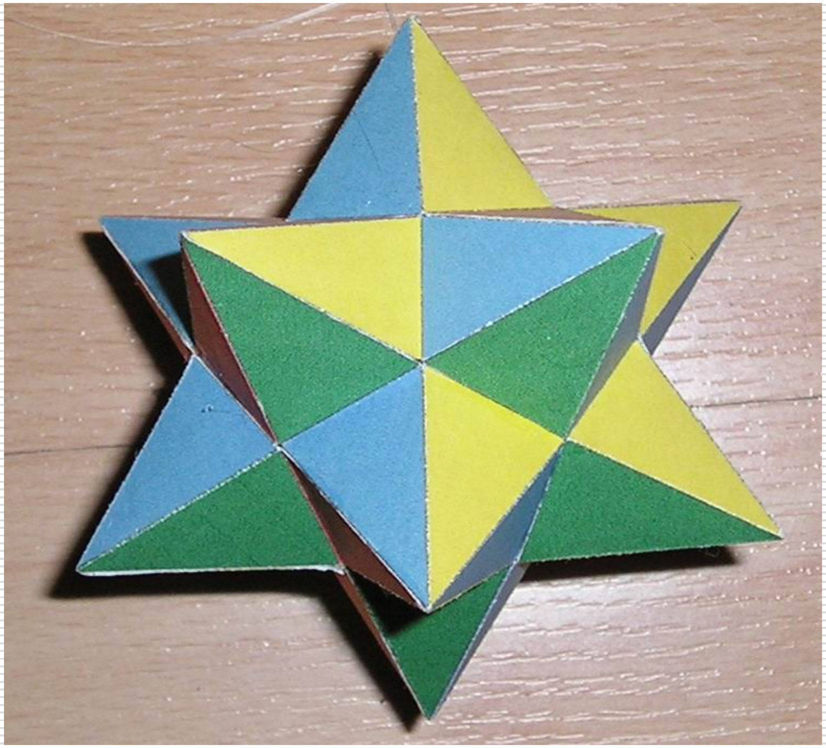
Great Dodecahedron

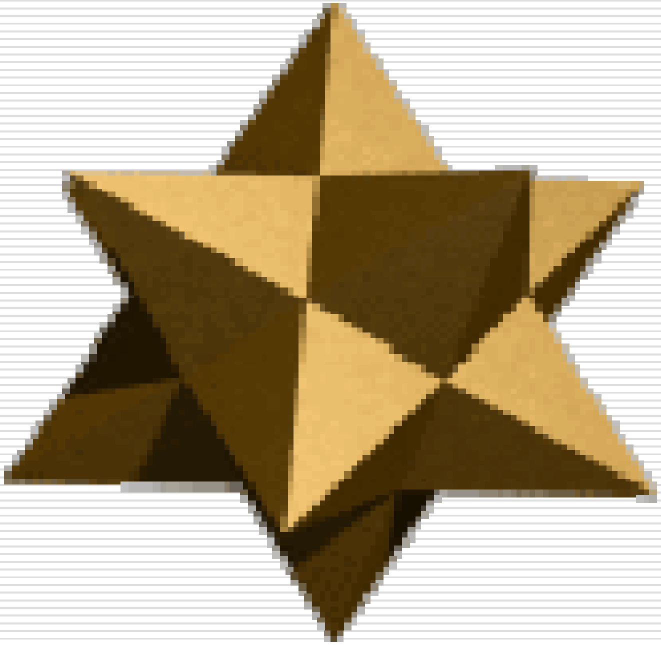




- Number of Faces: 12
- Number of Edges: 30 ➔ V-E+F=-6 (Euler Char.)
- Number of Vertices: 12

small stellated dodecahedron





Number of Faces: 12
Number of Edges: 30
Number of Vertices: 12

Euler Characteristic

- Let a closed surface have genus g. Then the polyhedral formula generalizes to the Poincaré formula

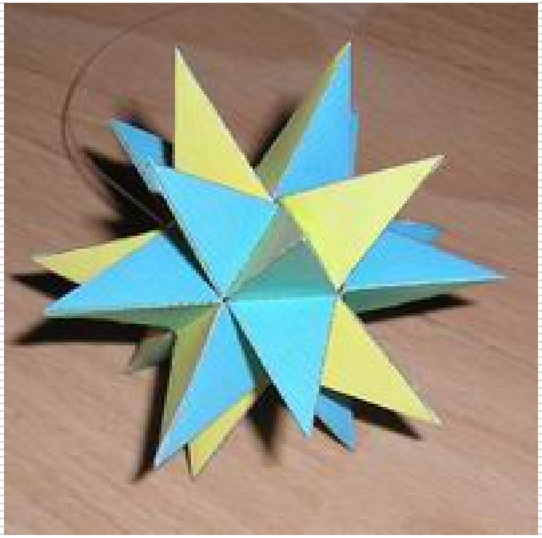
(1) $$\chi \equiv V - E + F = \chi(g),$$

- where

(2) $$\chi(g) = 2 - 2g$$

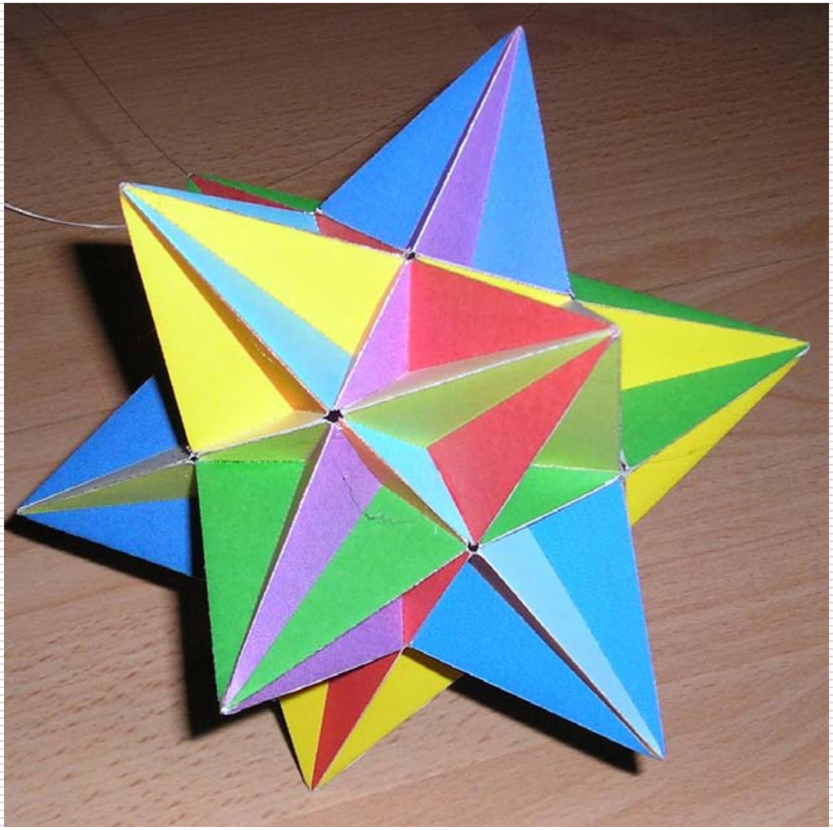
is the Euler characteristic, sometimes also known as the Euler-Poincaré characteristic. The polyhedral formula corresponds to the special case $g = 0$.

Great Stellated Dodecahedron



Number of Faces: 12
Number of Edges: 30
Number of Vertices: 20
Euler Char.=2

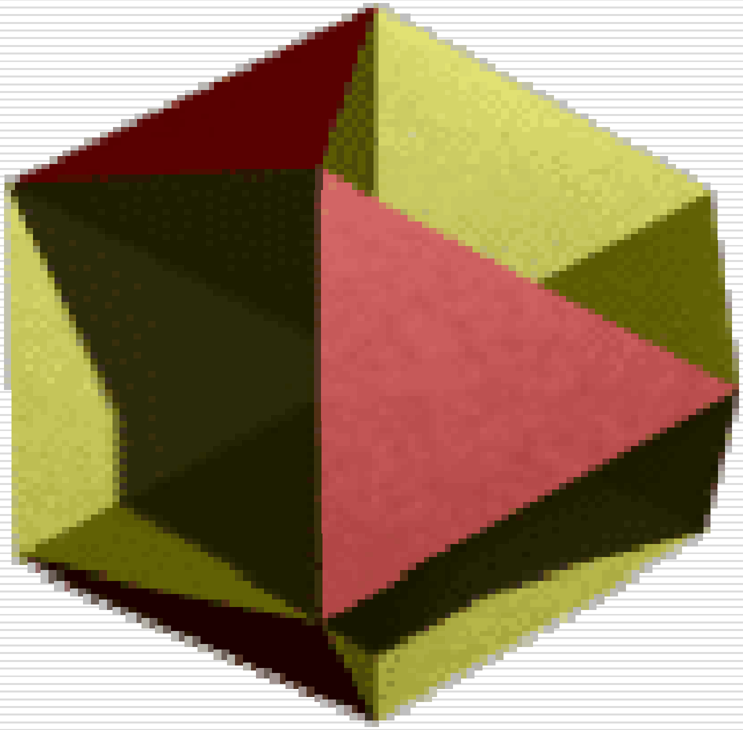
Great Icosahedron



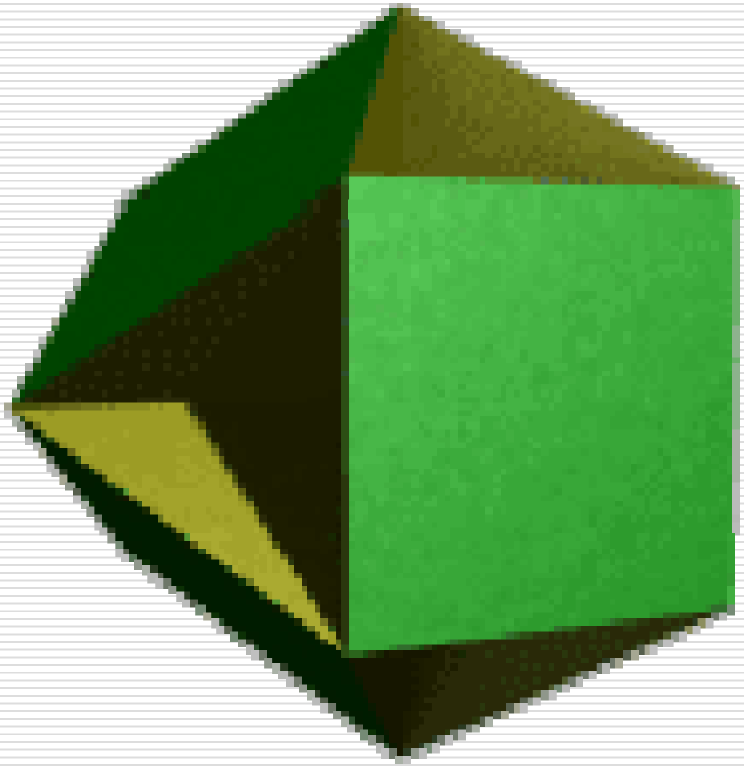
Great Icosahedron: 20
Number of Edges: 30
Number of Vertices: 12

Octahemioctahedron

- Number of Faces: 12
- Number of Edges: 24
- Number of Vertices: 12
- Euler Char.=0

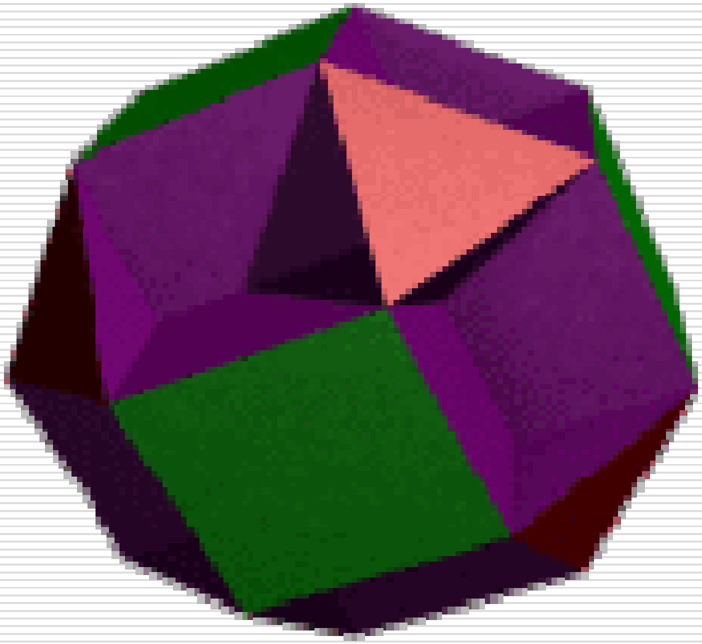


Cubohemioctahedron



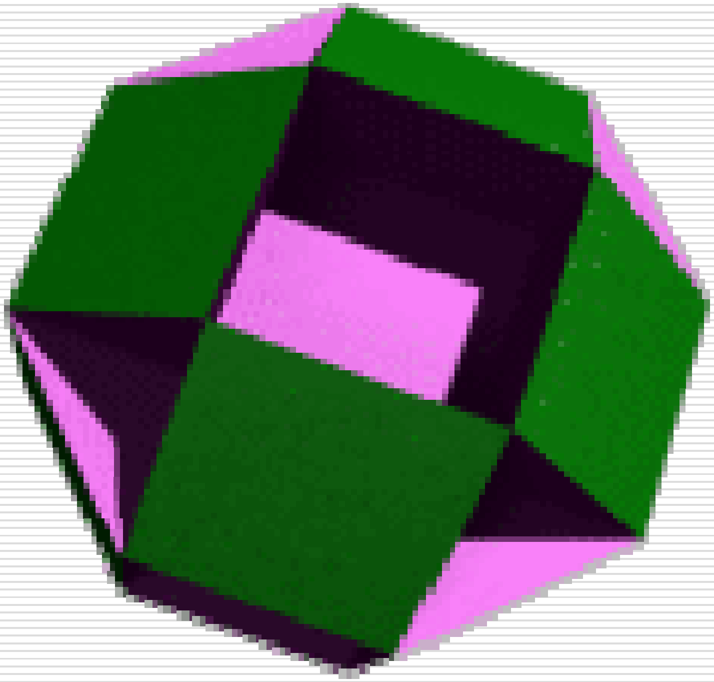
Number of Faces: 10
Number of Edges: 24
Number of Vertices: 12
Euler Char=-2

Small Cubicuboctahedron



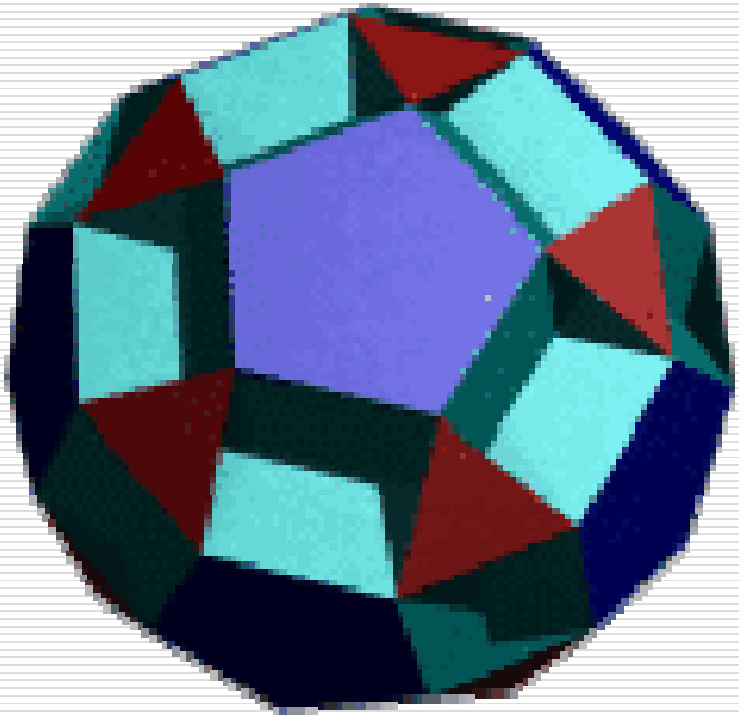
Number of Faces: 20
Number of Edges: 48
Number of Vertices: 24
Euler Char.= -4

Small Rhombihexahedron



Number of Faces: 18
Number of Edges: 48
Number of Vertices: 24
Euler Char.=-6

small dodecicosidodecahedron



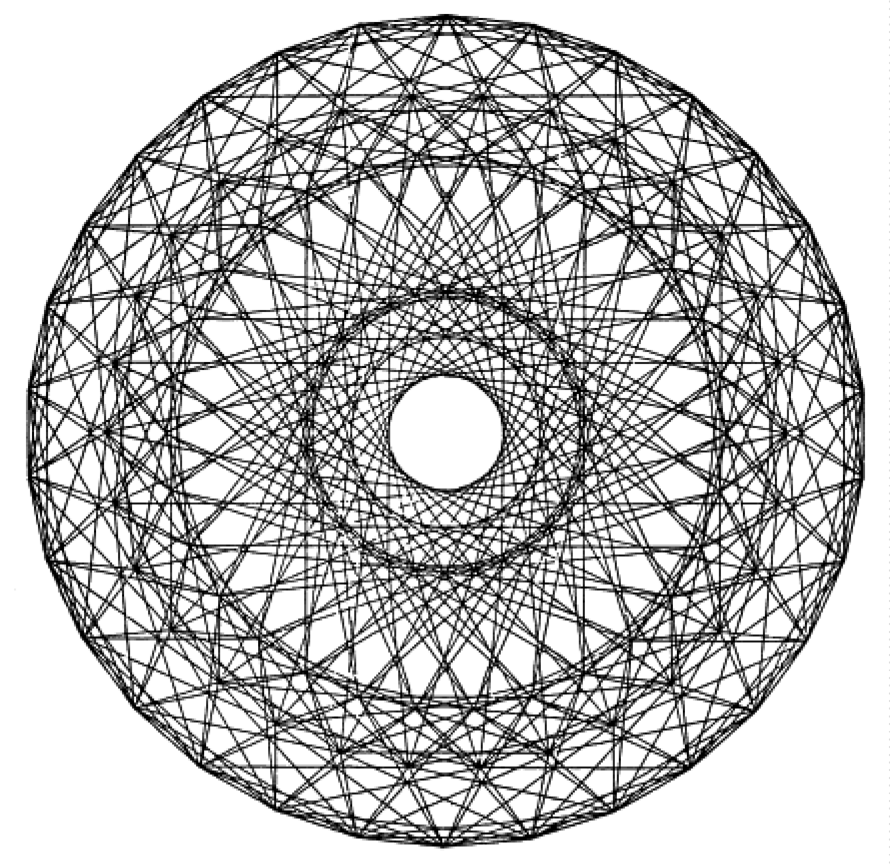
Number of Faces: 44
Number of Edges: 120
Number of Vertices: 60
Euler Char.=-16

Higher Dimensional Euler's Formula

- **$V - E + F - C = 0$**
- where
  - V = number of vertices
  - E = number of edges
  - F = number of faces
  - C = number of (3-dimensional) cells

hypericosahedron

- V = 120
- E = 720
- F = 1200
- C = 600



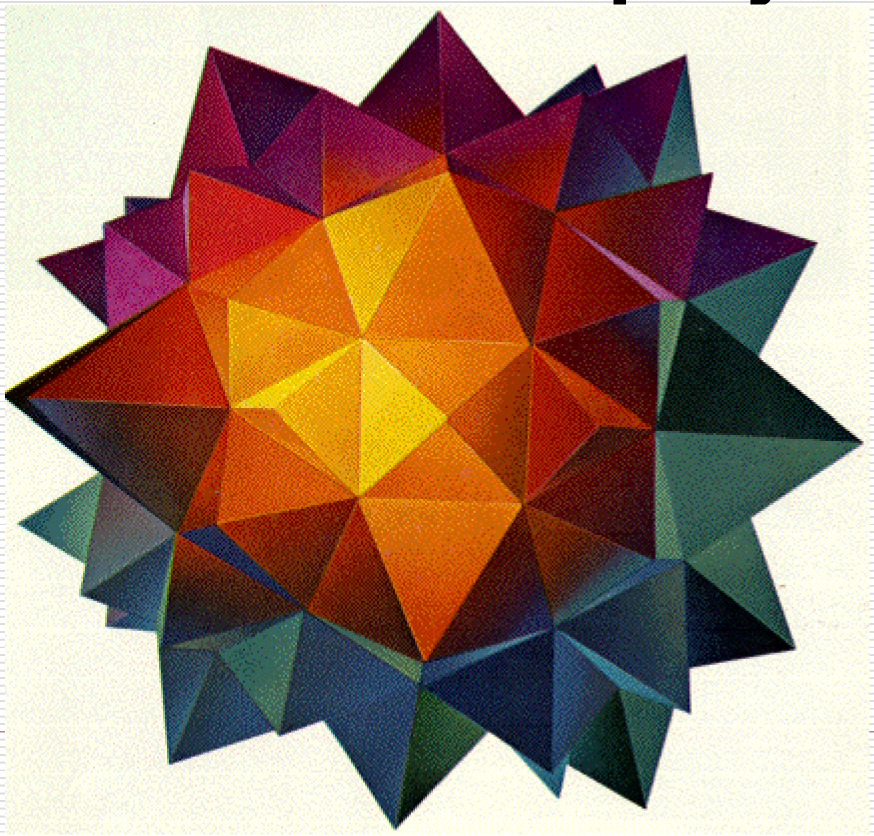
120 - 720 + 1200 - 600 = 0

hypericosahedron

- The above picture shows a 2-dimensional projection of the regular polyhedron in 4-dimensional Euclidean space with 600 tetrahedral cells, sometimes called a *hypericosahedron*.

Question

- How many faces, edges, and vertices in this polyhedron?



- Thank You !