(x2)

Calculus Test I 2009/07/08

1. (a) State the $(\varepsilon-\delta)$'s definition of limits

   (b) Use above (a) to prove $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$

2. Let $f(x) = x\left[\frac{1}{x}\right]$, where $[x]$ is the greatest integer functio.

   (a) Sketch the graph of $f(x)$ on the interval $\left[\frac{1}{4}, 2\right]$.

   (b) Show that for $x \neq 0$,
   $$\frac{1}{x} - 1 < \left[\frac{1}{x}\right] \leq \frac{1}{x}.$$
   Then use the Squeeze Theorem to prove that
   $$\lim_{x \to 0} x\left[\frac{1}{x}\right] = 1$$

3. (a) Prove : $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

   (b) Use above (a) to show
   $$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x \quad \forall x.$$

4. (a) Prove the following formulas :
   $$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$
   $$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
   $$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

   (b) Use the Riemann Sum and above (a) to show
   $$\int_0^1 x^2\,dx = \frac{1}{3} \quad \text{and} \quad \int_0^1 x^3\,dx = \frac{1}{4}$$

5. (a) Prove that : if f is differentiable at $x=c$, then f is continuous at $x=c$.

(b) Give an example to show that the reverse statement of above (a) is not true.

6. (a) Define $f(x)=\begin{cases} x\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$.

Show that $f(x)$ is continuous at $x=0$ but $f'(0)$ does not exist.

(b) Sketch the graph of $f(x)$.

7. Find the following $f'(x)$ respectively

(a) $f(x)=\dfrac{x^4+2x+1}{x+1}$ (b) $f(x)=\dfrac{\sin x}{x}$

(c) $f(x)=(\cos 6x+\sin x^2)^{\frac{1}{2}}$ (d) $f(x)=\sqrt{1+\sqrt{1+\sqrt{x}}}$

8. Find all points on the graph of $3x^2+4y^2+3xy=24$ where the tangent line is horizontal.

9. Find an equation of the tangent line at the given point.

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=2\ , \quad (1,1)$$

10. Calculate $F'(0)$, where $F(x)=\dfrac{x^9+x^8+4x^5-7x}{x^4-3x^2+2x+1}$

Test II

2009/07/15 (P1)

1. (a) If f is differentiable at $x=a$ and Δx is small, then state the following definition

   linear approximation of Δf

   linearization of $f(x)$ at $x=a$

   (b) Show that for any real number k,
   $$(1+x)^k \simeq 1+kx \text{ for small } x.$$
   Estimate $(1.02)^{0.7}$ and $(1.02)^{-0.3}$

2. (a) State the following definitions respectively.

   absolute minimum of $f(x)$ on an interval I

   absolute maximum of $f(x)$ 〃 〃 〃

   local minimum of $f(x)$, local maximum of $f(x)$

   critical point of $f(x)$

   (b) Find the maximum and minimum of $f(x)$ on I:

   (i) $f(x) = 1-(x-1)^{2/3}$, $I=[\frac{1}{2}, 2]$

   (ii) $f(x) = 2x^3-15x^2+24x+7$, $I=[0,6]$

※3. (a) State and prove the Mean Value Theorem (MVT)

   (b) Find a point c satisfying the conclusion of MVT for $f(x)$ on I

   (i) $f(x)=\sqrt{x}$, $I=[1,9]$

   (ii) $f(x)=\frac{x}{x+1}$, $I=[3,6]$

4 (a) State the following definitions respectively
concavity of a function; a point of inflection.

(b) State the following theorems respectively
First Derivative Test for Critical Points
Second Derivative Test

(c) Find the critical point, a point of inflection, extreme value, and sketch the graph of $f(x)$:

(i) $f(x) = \cos^2 x + \sin x$

(ii) $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$

5. Sketch the following graph of $f(x)$ respectively

(i) $f(x) = \frac{3x+2}{2x-4}$ (ii) $f(x) = 6x^7 - 7x^6$

*6 若一圓柱罐頭其容積固定為 V_0,
求製造此罐頭最省材料的造法,
並說明你的理由.

1 Calculate the follow sums : $\sum_{j=3}^{4} \sin(j \frac{\pi}{2})$

2. $\lim_{N \to \infty} \frac{\pi}{2N} \sum_{j=1}^{N} \sin(\frac{\pi}{3} + \frac{j\pi}{2N})$

3 Evalute $\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \sqrt{1-(\frac{j}{N})^2}$ by interpreting it as the area of part of a family geometric figure.

4 Prove that $0.277 \leq \int_{\pi/8}^{\pi/4} \cos x \, dx \leq 0.363$

5 $\int_{\pi/4}^{\pi/2} \frac{\sin x}{x} dx \leq \frac{\sqrt{2}}{2}$

6 $\int_0^5 |x^2-4x+3| dx$

7 What is the area (a positive number) between the x-axis and the graph of $f(x)$ on $[1, 3]$ if $f(x)$ is a negative function whose antiderivative F has the values $F(1)=7$ and $F(3)=4$

8 $\frac{d}{dx} \int_0^{x^2} \sin^2 t \, dt$

9 $\frac{d}{dx} \int_{x^2}^{x^4} \sqrt{t} \, dt$

10 Show that $\int_0^{\pi/6} f(\sin\theta) d\theta = \int_0^{1/2} f(u) \frac{1}{\sqrt{1-u^2}} du$

11 Sketch a region whose area is represented by $\int_{-\sqrt{2}/2}^{\sqrt{2}/2} (\sqrt{1-x^2} - |x|) dx$ and evaluate using geometry

8/4

(1) Find the following integrals respectively

(a) $\int_1^4 \sqrt{x}\ln x\,dx$ (b) $\int \frac{2x+5}{x^2+5x+6}\,dx$

(c) $\int \cos^3(\pi\theta)\sin^4(\pi\theta)\,d\theta$ (d) $\int \sin^4 x\cos^6 x\,dx$

(e) $\int \frac{\tan^3(\ln t)}{t}\,dt$ (f) $\int_0^{\frac{\pi}{3}} \tan x\,dx$

(g) $\int_0^{\frac{\pi}{4}} \tan^5 x\,dx$ (h) $\int \frac{1}{x^2\sqrt{x^2-2}}\,dx$

(i) $\int \frac{1}{\sqrt{x^2-4x+8}}\,dx$ (j) $\int \frac{x^2}{(x^2+1)^{3/2}}\,dx$

(k) $\int_0^2 xe^{9x}\,dx$ (l) $\int x^2\sin x\,dx$

(m) $\int e^x\cos x\,dx$ (n) $\int_0^1 \sqrt{1-x^2}\,dx$

(o) $\int_1^3 \ln x\,dx$

(2) Find the arc length of $f(x)=x^2$ from $x=0$ to $x=1$

(3) Set $I_m=\int_0^{\pi/2}\sin^m x\,dx$

(a) Show that $I_1=1$ $I_2=(\frac{1}{2})(\frac{\pi}{2})$ and prove that for $m>1$ $I_m=(\frac{m-1}{m})I_{m-2}$

(b) Show that $I_3=\frac{2}{3}$ and $I_4=(\frac{3}{4})(\frac{1}{2})(\frac{\pi}{2})$

(c) Show more generally:

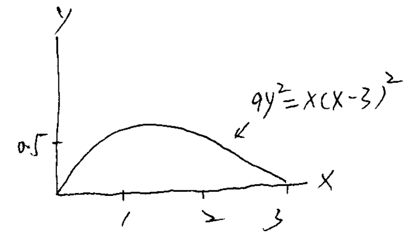
$$I_{2m}=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

$$I_{2m+1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\frac{2}{3}$$

(d) Conclude that

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2m\cdot 2m}{(2m-1)(2m+1)}\cdot\frac{I_{2m}}{I_{2m+1}}$$

(4) Find the arc length of the curve shown in



(5) Find the arc length of $y=\frac{1}{3}x^{3/2}-x^{1/2}$ $[2,8]$