Calculus : Final Exam.

1 (a) Prove : $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

(b) Use above (a) to show

$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x \quad \forall x$

2 (a) Define $f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Show that $f(x)$ is continuous at $x=0$ but $f'(0)$ does not exist.

(b) Sketch the graph of $f(x)$

3 (a) State and prove the Mean Value Theorem (MVT)

(b) Find a point c satisfying the conclusion of MVT for $f(x)$ on I

(i) $f(x) = \sqrt{x} \qquad I = [1, 9]$

(ii) $f(x) = \frac{x}{x+1} \qquad I = [3, 6]$

4 (a) State the ($\varepsilon - \delta$) definition of Limit

(b) Use (a) to prove : $\lim_{x\to 3} \frac{5}{x^2} = \frac{5}{9}$.

5 Show that if f is a quadratic polynomial, then the midpoint $c = \frac{a+b}{2}$ satisfies the conclusion of MVT on $[a, b]$

6 Let $f(x) = x\left[\frac{1}{x}\right]$ where $[x]$ is the greatest integer function

(a) Sketch the graph of $f(x)$ on $[\frac{1}{8}, 4]$

(b) Show $\lim_{x\to 0} x\left[\frac{1}{x}\right] = 1$

7 Show that $\int_0^{\pi/6} f(\sin\theta)\,d\theta = \int_0^{1/2} f(u)\frac{1}{\sqrt{1-u^2}}\,du$

8 $\frac{d}{dx}\int_{3x}^{x^2} \sin^2 t\,dt$

9 $\lim_{N\to\infty} \frac{\pi}{2N}\sum_{j=1}^{N} \sin\left(\frac{\pi}{3}+\frac{j\pi}{2N}\right)$

10 Find the following integrals respectively

(a) $\int \frac{2x+5}{x^2+5x+6}\,dx$

(b) $\int \cos^3(\pi\theta)\sin^4(\pi\theta)\,d\theta$

(c) $\int \frac{1}{\sqrt{x^2-4x+8}}\,dx$

(d) $\int e^x \cos x\,dx$

(e) $\int \frac{\tan^3(\ln t)}{t}\,dt$

11 (i) Find the arc length of $y=\frac{1}{3}x^{3/2}-x^{1/2}$, $[2, 8]$

(ii) Find the arc length of $y=3x^2$, $[0,2]$

12 Set $I_m=\int_0^{\pi/2} \sin^m x\,dx$

(a) Show that $I_1=1$ $I_2=\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$ and prove that for $m>1$ $I_m=\left(\frac{m-1}{m}\right)I_{m-2}$

(b) Show that $I_3=\frac{2}{3}$ and $I_4=\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$

(c) Show more generally :

$$I_{2m}=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

$$I_{2m+1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\cdots\frac{2}{3}$$

(d) Conclude that

$$\frac{\pi}{2}=\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\cdots\frac{2m\cdot 2m}{(2m-1)(2m+1)}\frac{I_{2m}}{I_{2m+1}}$$