Calculus M.-T. Exam. 2009/07/20 P1

1. (a) State the $(\varepsilon-\delta)$'s definition of limit.

   (b) Use (a) to prove $\lim_{x\to 3} \frac{1}{x^2} = \frac{1}{9}$

2. (a) State and Prove the Mean Value Theorem (MVT)

   (b) Show that if f is a quadratic polynomial, then the midpoint $c=\frac{a+b}{2}$ satisfies the conclusion of MVT on $[a,b]$ for any a and b.

3. Let $f(x) = x\left[\frac{1}{x}\right]$, where $[x]$ is the greatest integer function.

   (a) Sketch the graph of $f(x)$ on $[\frac{1}{8}, 4]$

   (b) Show $\lim_{x\to 0} x\left[\frac{1}{x}\right] = 1$

4. (a) Prove: $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

   (b) Use (a) to show: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

   $\forall\, x \in \mathbb{R}$, $(\tan x)' = \sec^2 x$, $(\cot x)' = -\csc^2 x$

   $(\sec x)' = \sec x \tan x$

   $(\csc x)' = -\csc x \cdot \cot x$

5. (a) State and Prove "First Derivative Test for Critical Points"

   (b) Let $f(x) = \cos^2 x + \sin x$.

   Find $\underset{0\le x\le \pi}{\text{Max}}\, f(x)$, $\underset{0\le x\le \pi}{\text{Min}}\, f(x)$, Sketch the graph of f.

6. Define $f(x) = x^3 \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$

(a) Find the critical points of $f(x)$ and show that $f'(x)$ is continuous at $x = 0$.

(b) Sketch the graph of $f(x)$ and $f'(x)$. Can the First Derivative Test be applied? Why?

(c) Show that $f(0)$ is neither a local min nor max.

7. Find an equation of the tangent line at the given point:

$$x^{\frac{1}{3}} + y^{\frac{1}{3}} = 2; \quad (1, 1)$$

8. Find $f'(x)$: (a) $f(x) = \sqrt{1+\sqrt{1+\sqrt{5x}}}$

(b) $f(x) = (\sin 6x + \cos x^3)^{\frac{3}{2}}$

(c) $f(x) = \left(\frac{x^4+2x+1}{x+1}\right)^{50}$

9. Sketch the following graph:

(a) $f(x) = \frac{3x+2}{2x-4}$ (b) $f(x) = ax^3 + bx^2 + cx + 1$

10. Find the following indefinite integrals:

(a) $\int [\sin(2t-9) + 20\cos 3t]\,dt$ (b) $\int \frac{x^2+2x-3}{x^4}\,dx$