1. Find the following $f'(x)$:

20%
(5%×4)

(a) $f(x)=\dfrac{\sin x^2}{1+x^2}$ (b) $f(x)=\sqrt{x+\sqrt{5x^2+1}}$

(c) $f(x)=\left(\cos 6x+\sin(x^3+1)\right)^{\frac{5}{2}}$ (d) $f(x)=x^{\frac{3}{2}}(2x^4+x^{-\frac{1}{2}})$

2. Find the local maximum, local minimum, the point of inflection of $f(x)$ if any one of them exists. Furthermore sketch the graph of $f(x)$. 图2分.

15%
(5%×3)

(i) $f(x)=3x^4-8x^3+6x^2+1$

(ii) $f(x)=\cos^2 x+\sin x$ (iii) $f(x)=\dfrac{x^2}{x-1}$

3. Solve the equation to get $y(t)$:

12%
(4%×3)

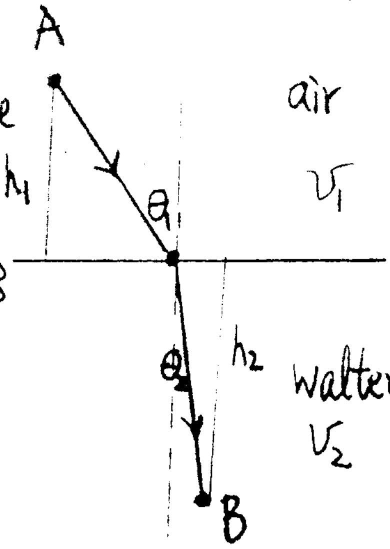
(i) $\dfrac{dy}{dt}=\cos(\pi t)$, $y(\frac{1}{2})=3$

(ii) $y''(t)=t-\cos t$, $y'(0)=2$, $y(0)=2$

(iii) $\dfrac{dy}{dt}=8t^3+3t^2-3$, $y(1)=1$

4. Design a cylindrical can of volume $20\,ft^3$ so that it uses the least amount of metal. In other words, minimize the surface area of the can.

10%




5. (Snell's Law) Right figure represents the surface of a swimming pool. A light beam travels from point A located above the pool to point B located undermeath the walter.
%

Let $V_1$: the velocity of light in air

$V_2$: " " " in walter

Prove Snell's Law of Refraction according to which the path from A to B that takes the least time satisfies:

$$\frac{\sin\theta_1}{V_1} = \frac{\sin\theta_2}{V_2}$$

6. Find the following indefinite integrals:
6%
(%×4) (i) $\int \frac{x^2+2x-3}{x^4}\,dx$ (ii) $\int (\sin(5t+3)+3\cos 3t)\,dt$

(iii) $\int \sec^2(3x)\,dx$ (iv) $\int (4t-9)^{-3}\,dt$

7. (a) Prove: $\lim_{h\to 0}\frac{\sin h}{h} = 1$, 5% and $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ 4%
5%

(b) Use (a) to show: $(\sin x)' = \cos x$, 3% $(\cos x)' = -\sin x$, 3% $\forall x$.

8. (a) Prove the formulas: $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$
2%
(6%×2) $1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$

(b) Use (a) and the Riemann Sum to show

$$\int_0^1 x^2\,dx = \frac{1}{3}, \quad \int_0^1 x^3\,dx = \frac{1}{4}$$

# 國立中央大學考試試卷
# National Central University Examination Answer Sheet

| | | | |
|---|---|---|---|
| ☐ 平時考(Quiz) | ☑ 期中考(Midterm) | ☐ 期末考(Final) | 評分 (Score) |
| 科目 (Course Title) | 日期 (Date) 7/20 | | |
| 系/級 (Department / Grade) | 班/組 (Class) | | |
| 學號 (Student ID) | 姓名 (Name) | | |

1.

(a) $f'(x) = \dfrac{2x\cdot\cos x^2\cdot(1+x^2) - 2x\cdot\sin x^2}{(1+x^2)^2} = \dfrac{2x\cdot[(1+x^2)\cdot\cos x^2 - \sin x^2]}{(1+x^2)^2}$

(b) $f'(x) = \frac{1}{2}\left[x+\sqrt{5x^2+1}\right]^{-\frac{1}{2}}\cdot\left(x+\sqrt{5x^2+1}\right)'$

$= \frac{1}{2}\left[x+\sqrt{5x^2+1}\right]^{-\frac{1}{2}}\cdot\left[1+\frac{1}{2}(5x^2+1)^{-\frac{1}{2}}\cdot 10x\right]$

$= \dfrac{1}{2\sqrt{x+\sqrt{5x^2+1}}}\left(1+\dfrac{5x}{\sqrt{5x^2+1}}\right)$

(c) $f'(x) = \frac{5}{2}\left[\cos 6x + \sin(x^3+1)\right]^{\frac{3}{2}}\cdot\left[\cos 6x + \sin(x^3+1)\right]'$

$= \frac{5}{2}\left[\cos 6x + \sin(x^3+1)\right]^{\frac{3}{2}}\left[-6\cdot\sin 6x + 3x^2\cdot\cos(x^3+1)\right]$

(d) $f(x) = 2x^{\frac{11}{2}} + x$

$\Rightarrow f'(x) = 11\cdot x^{\frac{9}{2}} + 1$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}}(2x^4 + x^{-\frac{1}{2}}) + x^{\frac{3}{2}}\cdot(8x^3 - \frac{1}{2}x^{-\frac{3}{2}})$

$= 3x^{\frac{9}{2}} + \frac{3}{2} + 8x^{\frac{9}{2}} - \frac{1}{2}$

$= 11\cdot x^{\frac{9}{2}} + 1$.

2.

(i) $f(x) = 3x^4 - 8x^3 + 6x^2 + 1$

$f'(x) = 12x(x-1)^2$

$f''(x) = 12(x-1)(3x-1)$

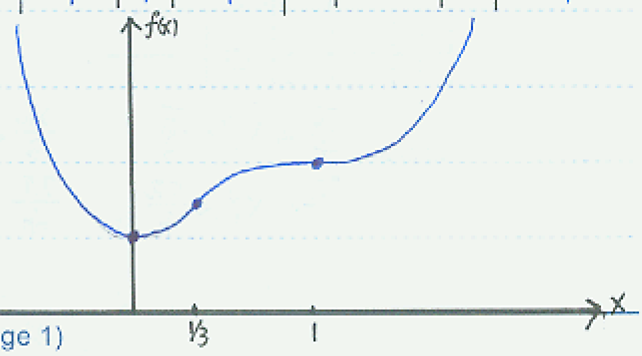
| $x$ | $(-\infty,0)$ | $0$ | $(0,\frac{1}{3})$ | $\frac{1}{3}$ | $(\frac{1}{3},1)$ | $1$ | $(1,\infty)$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | $0$ | $+$ | $+$ | $+$ | $0$ | $+$ |
| $f''(x)$ | $+$ | $+$ | $+$ | $0$ | $-$ | $0$ | $+$ |

$f'(x) = 0 \Rightarrow$ critical pts.: $x = 0, 1$

$f''(x) = 0 \Rightarrow x = \frac{1}{3}, 1$

point of inflection: $x = \frac{1}{3}, 1$

local min. $= f(0) = 1$



(ii) $f(x) = \cos^2 x + \sin x$

$f'(x) = -\cos x \cdot (2\cdot\sin x - 1)$

$f''(x) = 2\cdot\sin^2 x - 2\cdot\cos^2 x - \sin x$

$f'(x) = 0 \Rightarrow$ critical pts. : $x = \frac{\pi}{2} + n\pi,\ \frac{\pi}{6} + 2n\pi,\ \frac{5}{6}\pi + 2n\pi,\ n \in \mathbb{Z}$.

$f''(\frac{\pi}{2} + n\pi) > 0$ & $f(\frac{\pi}{2} + n\pi) = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd.} \end{cases}$

$f''(\frac{\pi}{6} + n\pi) < 0$ & $f(\frac{\pi}{6} + n\pi) = \frac{5}{4}$

$f''(\frac{5\pi}{6} + n\pi) < 0$ & $f(\frac{5}{6}\pi + n\pi) = \frac{5}{4}$

local min. $= f(\frac{\pi}{2} + n\pi) = -1$ where $n \in \mathbb{Z}$ and $n$ is odd.

local max. $= f(\frac{\pi}{6} + 2n\pi) = f(\frac{5}{6}\pi + 2n\pi) = \frac{5}{4}$, $n \in \mathbb{Z}$.

consider $[0, 2\pi]$ $f''(x) = 0 \Rightarrow 2\cdot\sin^2 x - 2\cdot\cos^2 x - \sin x = 0 \Rightarrow 4\cdot\sin^2 x - \sin x - 2 = 0$

$\Rightarrow \sin x = \frac{1 \pm \sqrt{33}}{8} \Rightarrow x = \sin^{-1}\frac{1 \pm \sqrt{33}}{8}$ note: $\sqrt{33} \approx 5.745$

case 1 : $x_1 = \sin^{-1}\frac{1+\sqrt{33}}{8} \approx \sin^{-1}\frac{6.745}{8} \approx \sin^{-1} 0.843$

$\Rightarrow \frac{\pi}{4} < x_1 < \frac{\pi}{3}$ or $\frac{2\pi}{3} < x_1 < \frac{3\pi}{4}$ $x_1$的值很接近 $\frac{\pi}{3}$ or $\frac{2\pi}{3}$

case 2 : $x_2 = \sin^{-1}\frac{1-\sqrt{33}}{8} \approx \sin^{-1}(-0.593)$

$\Rightarrow \frac{7\pi}{6} < x_2 < \frac{5\pi}{4}$ or $\frac{7\pi}{4} < x_2 < \frac{11\pi}{6}$ $x_2$的值很接近 $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$

consider $f(x)$ on $[0, 2\pi]$

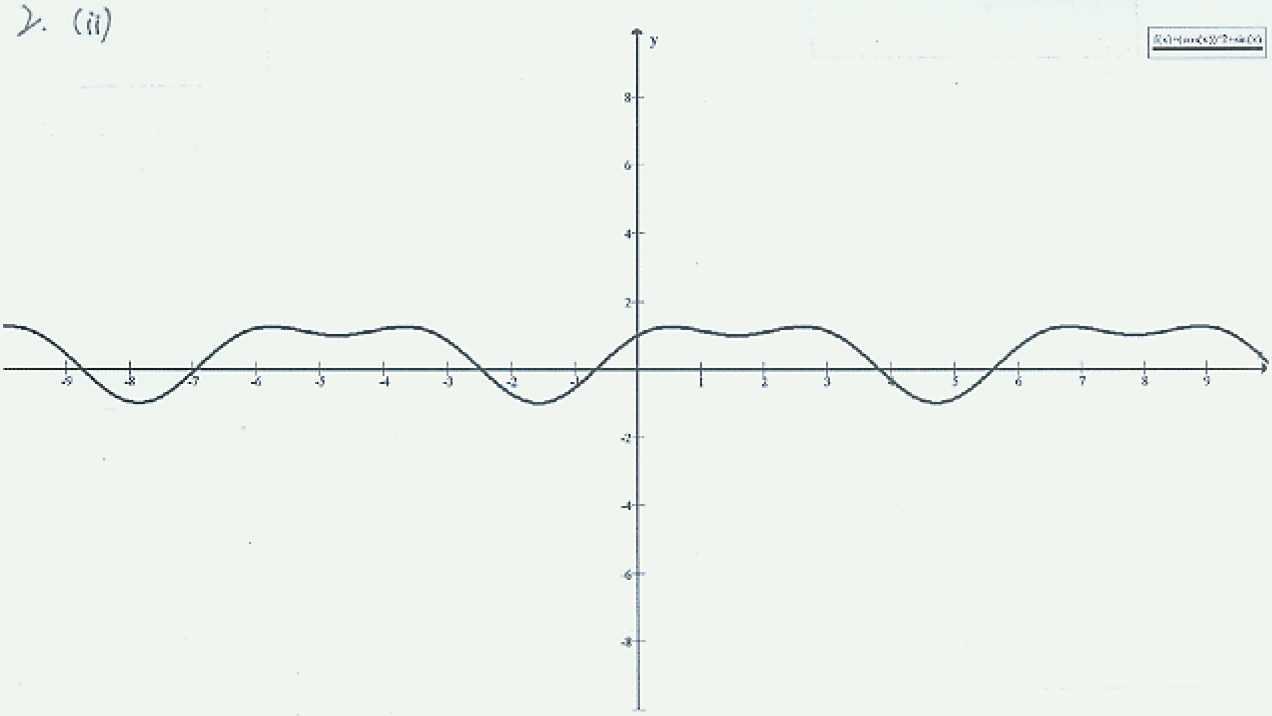
| $x$ | $[0, \frac{\pi}{6})$ | $\frac{\pi}{6}$ | $(\frac{\pi}{6}, x_1^{(1)})$ | $x_1^{(1)}$ | $(x_1^{(1)}, \frac{\pi}{2})$ | $\frac{\pi}{2}$ | $(\frac{\pi}{2}, x_1^{(2)})$ | $x_1^{(2)}$ | $(x_1^{(2)}, \frac{5\pi}{6})$ | $\frac{5\pi}{6}$ | $(\frac{5\pi}{6}, x_2^{(1)})$ | $x_2^{(1)}$ | $(x_2^{(1)}, \frac{3\pi}{2})$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | + | 0 | − | − | − | 0 | + | + | + | 0 | − | − | − |
| $f''(x)$ | − | − | − | 0 | + | + | + | 0 | − | − | − | 0 | + |

| $x$ | $\frac{3\pi}{2}$ | $(\frac{3\pi}{2}, x_2^{(2)})$ | $x_2^{(2)}$ | $(x_2^{(2)}, 2\pi]$ |
|---|---|---|---|---|
| $f'(x)$ | 0 | + | + | + |
| $f''(x)$ | + | + | 0 | − |

where $\frac{\pi}{4} < x_1^{(1)} < \frac{\pi}{3}$, $\frac{2\pi}{3} < x_1^{(2)} < \frac{3\pi}{4}$

$\frac{7\pi}{6} < x_2^{(1)} < \frac{5\pi}{4}$, $\frac{7\pi}{4} < x_2^{(2)} < \frac{11\pi}{6}$

2. (ii)



(iii) $f(x) = \dfrac{x^2}{x-1}$

$f'(x) = \dfrac{x(x-2)}{(x-1)^2}$ ; $f''(x) = \dfrac{2}{(x-1)^3}$

1° $f'(x) = 0 \Rightarrow$ critical pts. : $x = 0, 2$ 　 Note: $x=1$ is not a critical pt. Since $f(x)$ is not defined at $x=1$

A point of inflection of $f(x)$ does not exist.

2° $\because$ $f''(x) > 0$ on $x > 1$ and $f''(x) < 0$ on $x < 1$

$\Rightarrow f''(0) < 0$ and $f''(2) > 0$

$\therefore$ $f(0) = 0$ is the local maximum

$f(2) = 4$ is the local minimum.

| $x$ | $(-\infty, 0)$ | $0$ | $(0,1)$ | $(1,2)$ | $2$ | $(2,\infty)$ |
|---|---|---|---|---|---|---|
| $f'(x)$ | + | 0 | − | + | 0 | − |
| $f''(x)$ | − | − | − | + | + | + |

$\lim_{x\to 1^+} f(x) = +\infty$ , $\lim_{x\to 1^-} f(x) = -\infty$

$\lim_{x\to +\infty} f(x) = +\infty$ , $\lim_{x\to -\infty} f(x) = -\infty$

(a) $x = 1$ is a vertical

(b) $\because$ $\lim_{x\to+\infty} f'(x) = \lim_{x\to-\infty} f'(x) = 1$

$\therefore$ there exists a slant asymptote of $f(x)$

Let $y = mx + b$ be a slant asymptote of $f(x)$

$$\Rightarrow \lim_{x\to\pm\infty}\left[\frac{x^2}{x-1}-(mx+b)\right]=0$$

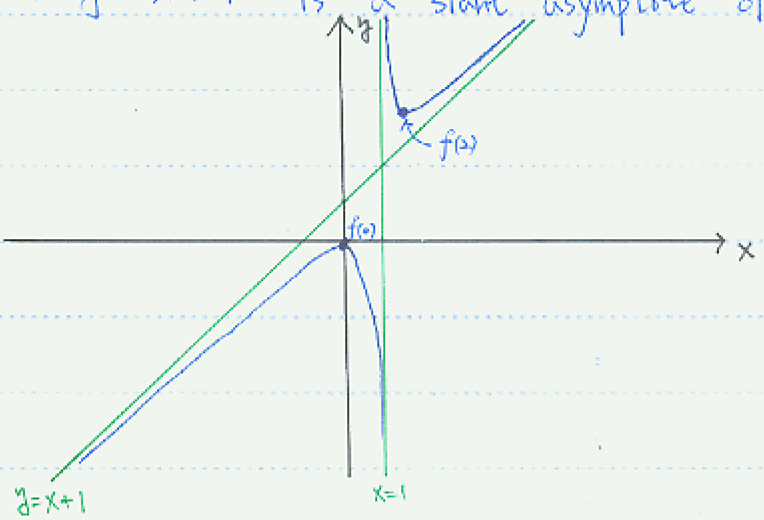
$$\Rightarrow \lim_{x\to\pm\infty} x\left(\frac{x}{x-1}-m-\frac{b}{x}\right)=0$$

$$\Rightarrow \lim_{x\to\pm\infty} \frac{x}{x-1}-m-\frac{b}{x}=0$$

$$\Rightarrow m=1 \qquad \text{i.e. } y=x+b$$

$$\because \lim_{x\to\pm\infty}\left[\frac{x^2}{x-1}-(x+b)\right]=0 \Rightarrow \lim_{x\to\pm\infty}\frac{x^2-x(x-1)}{x-1}-b=0 \Rightarrow b=1$$

$\therefore$ $y=x+1$ is a slant asymptote of $f(x)$




3.

(i) $y(t)=\int \cos\pi t\, dt = \frac{1}{\pi}\sin\pi t + C$

$y(\frac{1}{2}) = \frac{1}{\pi}\cdot\sin\frac{\pi}{2} + C = 3 \Rightarrow C = 3-\frac{1}{\pi}$

$\therefore\ y(t)=\frac{1}{\pi}\sin\pi t + 3 - \frac{1}{\pi}$

(ii) $y'(t)=\int t-\cos t\, dt = \frac{1}{2}t^2-\sin t + C_1$

$y'(0) = C_1 = 2 \Rightarrow y'(t)=\frac{1}{2}t^2-\sin t+2.$

$y(t)=\int \frac{1}{2}t^2-\sin t+2\, dt = \frac{1}{6}t^3+\cos t+2t+C_2$

$y(0) = 1+C_2 = 2 \Rightarrow C_2 = 1$

$\therefore\ y(t)=\frac{1}{6}t^3+2t+\cos t+1$

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(iii) $y(t) = \int 8t^3 + 3t^2 - 3 \, dt = 2t^4 + t^3 - 3t + C$

$y(1) = 2 + 1 - 3 + C = 1 \Rightarrow C = 1$

$\therefore y(t) = 2t^4 + t^3 - 3t + 1$

4.

$V = \pi r^2 h = 20 \quad \Rightarrow h = \frac{20}{\pi r^2}$

$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{40}{r}$

$\Rightarrow \frac{d}{dr} S(r) = 4\pi r - \frac{40}{r^2}$

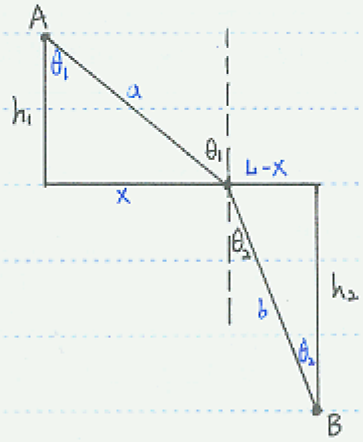
$\frac{d}{dr} S(r) = 0 \quad \Rightarrow r = \left(\frac{10}{\pi}\right)^{\frac{1}{3}}$

$\Rightarrow h = \frac{20}{\pi} \cdot r^{-2} = 2 \cdot \frac{10}{\pi} \cdot \left(\frac{10}{\pi}\right)^{-\frac{2}{3}} = 2\left(\frac{10}{\pi}\right)^{\frac{1}{3}} = 2r$

$\lim_{r \to 0} S(r) = \infty$

$\lim_{r \to \infty} S(r) = \infty$

5.




$$f(x) = \frac{a}{v_1} + \frac{b}{v_2} = \frac{\sqrt{x^2+h_1^2}}{v_1} + \frac{\sqrt{(L-x)^2+h_2^2}}{v_2}$$

$$\Rightarrow f'(x) = \frac{x}{v_1\sqrt{x^2+h_1^2}} - \frac{L-x}{v_2\sqrt{(L-x)^2+h_2^2}}$$

$$= \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

$$f''(x) = \frac{h_1^2}{v_1(x^2+h_1^2)^{3/2}} + \frac{h_2^2}{v_2[(L-x)^2+h_2^2]^{3/2}} > 0$$

$$f'(x) = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

6.

(i) $\int \frac{x^2+2x-3}{x^4}\,dx = -x^{-1} - x^{-2} + x^{-3} + C$

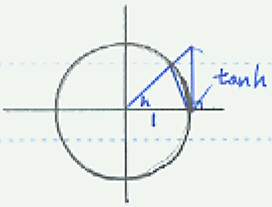
(ii) $\int \sin(5t+3) + 3\cdot\cos 3t\,dt = -\frac{1}{5}\cos(5t+3) + \sin 3t + C$

(iii) $\int \sec^2(3x)\,dx = \frac{1}{3}\tan(3x) + C$

(iv) $\int (4t-9)^{-3}\,dt = -\frac{1}{8}(4t-9)^{-2} + C$

7.

(a) i) $\frac{1}{2}\sin h \le \frac{1}{2}h \le \frac{1}{2}\frac{\sin h}{\cos h}$, $0 < h < \frac{\pi}{2}$.




$\because \frac{1}{2}\sin h \le \frac{1}{2}h \Rightarrow \frac{\sin h}{h} \le 1$ and

$\frac{1}{2}h \le \frac{1}{2}\frac{\sin h}{\cos h} \Rightarrow \cos h \le \frac{\sin h}{h}$

$\therefore \cos h \le \frac{\sin h}{h} \le 1$, for $0 < h < \frac{\pi}{2}$.

$\because \lim_{h\to 0^+} \cos h = \lim_{h\to 0^+} 1 = 1$

by Squeeze Thm.: $\lim_{h\to 0^+} \frac{\sin h}{h} = 1$

Because $\cosh$ and $\frac{\sinh}{h}$ are even function, we have the same result for $-\frac{\pi}{2} < h < 0$.

$\cos h \leq \frac{\sin h}{h} \leq 1$ , for $-\frac{\pi}{2} < h < 0$

$\Rightarrow \lim_{h \to 0^-} \frac{\sin h}{h} = 1$ (by Squeeze Thm.)

Hence $\lim_{h \to 0} \frac{\sin h}{h} = 1$

2) $\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h \to 0} \frac{-\sin^2 h}{h(\cos h + 1)} = \lim_{h \to 0} \frac{\sin h}{h} \frac{-\sin h}{\cos h + 1}$

$\underset{(\text{by } 1)}{=} 0$

(b) $(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h}$

$= \lim_{h \to 0} \left( \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right)$

$= \cos x$

$(\cos x)' = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cdot \cos h - \sin x \cdot \sin h - \cos x}{h}$

$= \lim_{h \to 0} \left( \cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \right)$

$= -\sin x$

8.

(a) $\because (1+k)^3 = 1 + 3 \cdot k + 3 k^2 + k^3$

$\Rightarrow (1+1)^3 = 1 + 3 \cdot 1 + 3 \cdot 1^2 + 1^3$

$(1+2)^3 = 1 + 3 \cdot 2 + 3 \cdot 2^2 + 2^3$

$(1+3)^3 = 1 + 3 \cdot 3 + 3 \cdot 3^2 + 3^3$

$\vdots \qquad \vdots$

+) $(1+n)^3 = 1 + 3 \cdot n + 3 \cdot n^2 + n^3$

$(1+n)^3 = n + 3 \cdot \sum_{k=1}^{n} k + 3 \cdot \sum_{k=1}^{n} k^2 + 1 \quad \Rightarrow \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$\because\ (1+k)^4 = 1 + 4\cdot k + 6\cdot k^2 + 4\cdot k^3 + k^4$

$\Rightarrow\ (1+1)^4 = 1 + 4\cdot 1 + 6\cdot 1^2 + 4\cdot 1^3 + 1^4$

$(1+2)^4 = 1 + 4\cdot 2 + 6\cdot 2^2 + 4\cdot 2^3 + 2^4$

$(1+3)^4 = 1 + 4\cdot 3 + 6\cdot 3^2 + 4\cdot 3^3 + 3^4$

$\vdots \qquad\qquad \vdots$

$+)\ \ (1+n)^4 = 1 + 4\cdot n + 6\cdot n^2 + 4\cdot n^3 + n^3$

$(1+n)^4 = n + 4\cdot\sum_{k=1}^{n} k + 6\cdot\sum_{k=1}^{n} k^2 + 4\cdot\sum_{k=1}^{n} k^3 + 1 \Rightarrow \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

(b) Let $P = \{0 = x_0 < x_1 < \dots < x_n = 1\}$ be a partition of $[0,1]$ and $\Delta x_k = \frac{1}{n}$

1° $R_n = \sum_{k=1}^{n}\left(\underset{[x_{k-1},x_k]}{\mathrm{Max}} f(x)\right)\cdot\Delta x_k = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} = \sum_{k=1}^{n}\frac{k^2}{n^3} = \frac{n(n+1)(2n+1)}{6n^3}$

$\Rightarrow \lim_{n\to\infty} R_n = \frac{1}{3}$

$L_n = \sum_{k=1}^{n}\left(\underset{[x_{k-1},x_k]}{\mathrm{Min}} f(x)\right)\cdot\Delta x_k = \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} = \sum_{k=0}^{n-1}\frac{k^2}{n^3} = \frac{n(n-1)(2n-1)}{6n^3}$

$\Rightarrow \lim_{n\to\infty} L_n = \frac{1}{3}$

$\because\ L_n \le \sum_{k=1}^{n} f(c_k)\cdot\Delta x_k \le R_n$ where $c_k \in [x_{k-1}, x_k]$

$\Rightarrow \int_0^1 x^2\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\cdot\Delta x = \frac{1}{3}$

2° $R_n = \sum_{k=1}^{n}\left(\underset{[x_{k-1},x_k]}{\mathrm{Max}} f(x)\right)\cdot\Delta x_k = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} = \sum_{k=1}^{n}\frac{k^3}{n^4} = \left(\frac{n(n+1)}{2n^2}\right)^2$

$\Rightarrow \lim_{n\to\infty} R_n = \frac{1}{4}$

$L_n = \sum_{k=1}^{n}\left(\underset{[x_{k-1},x_k]}{\mathrm{Min}} f(x)\right)\cdot\Delta x_k = \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} = \sum_{k=0}^{n-1}\frac{k^3}{n^4} = \left(\frac{n(n-1)}{2n^2}\right)^2$

$\Rightarrow \lim_{n\to\infty} L_n = \frac{1}{4}$

$\because\ L_n \le \sum_{k=1}^{n} f(c_k)\cdot\Delta x_k \le R_n$ where $c_k \in [x_{k-1}, x_k]$

$\Rightarrow \int_0^1 x^3\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k = \frac{1}{4}$