Calculus II : Test 1   2009/08/17   P1

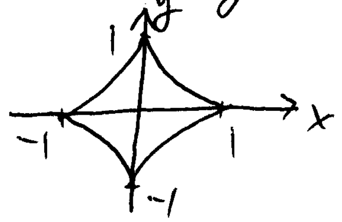
1. (i) If $f'(x)$ exists and is continuous on $[a,]$, then prove

$$\text{Arc length over } [a,b] = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

(Arc length $= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(x_i-x_{i-1})^2+(f(x_i)-f(x_{i-1}))}$ by mean value theorem)

(ii) Using above (i) to find the arc length of $f(x)=x^2$ over $[0,1]$.   ($\int \sec^3\theta$)

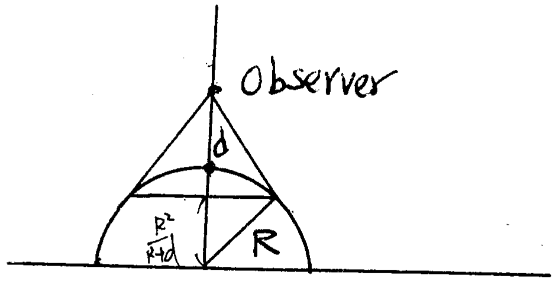
2. (i) Calculate the length of the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$



($4\int_0^1 \sqrt{1+(y')^2}$)

(ii) Show that the circumference of the unit circle is equal to $\left(2\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}\right)$. Evaluate, thus verifying that the circumference is 2π.

3. Prove that the portion of a sphere of radius R seen by an observer located at a distance d above the North Pole has area $A=\dfrac{2\pi d R^2}{d+R}$.



($\int_{\frac{R^2}{R+d}}^{R} 2f(x)\sqrt{1+(f'(x))^2}\,\pi\,dx$)

$f'(x)=\frac{-x}{\sqrt{R^2-x^2}}$

$y^2+x^2=R^2$

$\to y=\sqrt{R^2-x^2}$

4 (i) Let $f^{(n+1)}(x)$ exists and be continuous $\forall x$.
Find the nth Taylor polynomial $T_n(x)$ for f centered at $x=a$, and prove

$$R_n(x) = f(x) - T_n(x) = \frac{1}{n!}\int_a^x (x-u)^n f^{(n+1)}(u)\,du.$$

(ii) Let $\begin{cases} f(x) = \sin x. \\ a = 0. \end{cases}$ Show that the Taylor polynomia for f are

$$T_{2n-1}(x) = T_{2n}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

Use the Error bound with $n=4$ to show

$$\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \le \frac{|x|^5}{120} \quad \forall x.$$

5 A cylindrical tank of heigh 9 ft and radius 2 ft is filled with walter. Walter drains through a square hole of side 1 inch in the bottom. Determine the walter level $y(t)$ at time t (seconds).
How long does it take for the tank to go from full to empty.

6. (a) Solve the logistic differential equation

$$\frac{dy}{dt} = ky\left(1-\frac{y}{A}\right), \quad y(0) = y_0$$

(b) Solve the differential equation
(i) $\sqrt{1-x^2}\, y' = y^2, \quad y(0) = 1$
(ii) $y' + x^{-1}y = \cos(x^2)$.

Calculus Test II 2009/08/31

1. Find the range and radius of convergence :

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(a) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^2}$ P.585 ex 6 $[-1,1]$ (b) $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{\ln n}$ P.585 ex 11 $[-1,1)$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(x-5)^n$ P.578 example 2 $(4,6]$

(d) $\displaystyle\sum_{n=2}^{\infty}\frac{(x-2)^n}{(n\ln n)^2}$ P.586 ex 26 $[1,3]$ (e) $\displaystyle\sum_{n=10}^{\infty} n!\,(x+5)^n$ P.586 ex 21 $x=-5$

2 Find a power series solution to the differential Eq.

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(i) $\begin{cases} x^2y''+xy'+(x^2-1)y=0 \\ y'(0)=2 \end{cases}$

P.582 example 8

$P(x)=\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k\cdot 2}{4^k k!\,(k+1)!}x^{2k+1}$

(ii) $\begin{cases} y''-xy'+y=0 \\ y(0)=1,\ y'(0)=0 \end{cases}$

P.586 ex 47 $P(x)=1-\frac{1}{2}x^2-\frac{1}{4!}x^4-\frac{3}{6!}x^6-\frac{15}{8!}x^8\ldots$

3 Let $J=\displaystyle\int_0^1 \sin(x^2)\,dx$ P.592 example 8

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(a) Express J as an infinite series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{4n+2}$

(b) Determine J to within an error less than 10^{-2}

$n=2$

4 Find the Taylor series of the following $f(x)$

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(i) $f(x)=e^{\sin x}$ $a=0$ P.598 ex 62

$1+x+\frac{1}{2}x^2-\frac{1}{8}x^4+\ldots$

(ii) $f(x)=\dfrac{\sin x}{x}$, $a=0$

5. (a) Find parametric equation for the cycloid generated by a point P on the unit circle

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P.608 example 7

$x(t)=t-\sin t$

$y(t)=1-\cos t$

(b) Calculate the length s of one arch of the cycloid generated by the unit circle.

$\displaystyle\int_0^{2\pi}\sqrt{(x'(t))^2+(y'(t))^2}\,dt=\int_0^{2\pi}\sqrt{(1-\cos t)^2+(\sin t)^2}\,dt$

$=8$

P618 example 3

6. A particle travels along the path

15% $c(t) = (2t,\ 1+t^{3/2})$, (t in minutes, distance in feet).

(a) Find the speed at $t=1$ $s'(t) = \sqrt{x'(t)+y'(t)} = \sqrt{4+\frac{9}{4}t}$ $t=1$
$= \sqrt{4+\frac{9}{4}}$ ft/min

(b) Compute distance traveled s and displacement d during the first 4 min.
$s = \int_0^4 \sqrt{4+\frac{9}{4}t}\,dt = \frac{8}{27}(13^{3/2}-8)$

7. The curve $c(t) = (t-\tanh t,\ \operatorname{sech} t)$ P619 example

10% is called a tractrix. Calculate the surface of the infinite surface generated by revolving the tractrix about the x-axis for

$0 \le t < \infty$.

$S = 2\pi\int_0^\infty \operatorname{sech}\sqrt{\tanh^2 t}\,dt$

$= 2\pi$