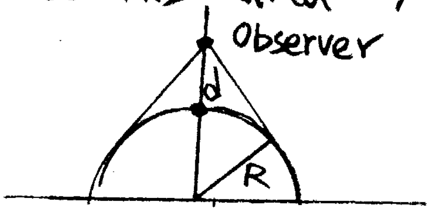
Calculus II : Test 1. 2009/08/17

1. (i) Find the arc length of $f(x) = x^2$ over $[1, 2]$.

   (ii) Calculate the length of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

2. Prove that the portion of a sphere of radius R seen by an observer located at a distance d above the North Pole has area $A = \dfrac{2\pi d R^2}{d+R}$



3. Let $T_n(x)$ be the Taylor polynomial for $f(x) = \ln x$ at $a=1$ and let $c>1$.

   (a) Show that the maximum of $f^{(k+1)}(x)$ on $[1, c]$ is $f^{(k+1)}(1)$.

   (b) Prove $\left| T_n(c) - \ln c \right| \leq \dfrac{|c-1|^{n+1}}{n+1}$

   (c) Find n such that $\left| T_n(1.5) - \ln 1.5 \right| \leq 10^{-2}$.

4 Solve the following initial value problem

(a) $\begin{cases} y' = -ty \\ y(0) = 3 \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = t \tan x \\ x(0) = 1 \end{cases}$

(c) $\begin{cases} y^2 \dfrac{dy}{dx} = x^{-3} \\ y(2) = 0 \end{cases}$ (d) $\begin{cases} \sqrt{1-x^2}\, y' = y^2 \\ y(0) = 1 \end{cases}$

5. A tank, in the following respectively, filled with water. Let $y(t)$ be the water level at time t.

(a) Find the differential equation satisfied by $y(t)$ and solve the $y(t)$

(b) How long does it take for the tank to empty.

6. Solve the logistic differential equation

$$\begin{cases} \dfrac{dy}{dt} = \alpha y\left(1-\dfrac{y}{A}\right), \quad \alpha, A : \text{positive constants} \\ y(t_0) = y_0 \end{cases}$$

Prove $\lim\limits_{t\to\infty} y(t) = A$.

7. Find the solution $y(x)$:

(a) $y' - (\ln x)y = x^x$

(b) $y' + \frac{1}{x+1}y = x^{-2}$, $y(1) = 2$

(c) $y' + (\sec x)y = \cos x$

(d) $(\sin x)y' - (\cos x)y = 1$, $y(\frac{\pi}{4}) = 0$

8. Find the following limit respectively

(a) $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$ (b) $\lim\limits_{n\to\infty} \left(1+\frac{1}{n^2}\right)^n$

①

國立中央大學考試試卷
National Central University Examination Answer Sheet

☑ 平時考(Quiz) ☐ 期中考(Midterm) ☐ 期末考(Final)		評分 (Score)
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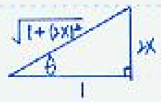
1.

(i) Arc length of $f(x)$ over $[1,2]$

$= \int_1^2 \sqrt{1+(2x)^2}\, dx$



$2x = \tan\theta \Rightarrow 2dx = \sec^2\theta\, d\theta$

$1+(2x)^2 = \sec^2\theta$

$= \frac{1}{2}\int_{x=1}^{x=2} \sec^3\theta\, d\theta$

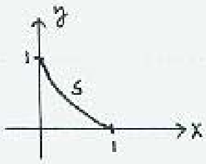
$\because \int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\cdot\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$

$\Rightarrow \int_{x=1}^{x=2}\sec^3\theta\, d\theta = \frac{1}{2}\sqrt{1+4x^2}\cdot 2x\Big|_{x=1}^{2} + \frac{1}{2}\ln\left|\sqrt{1+4x^2}+2x\right|\Big|_{x=1}^{2}$

$= 2\sqrt{17} - \sqrt{5} + \frac{1}{2}\left[\ln(\sqrt{17}+4) - \ln(\sqrt{5}+2)\right]$

$\therefore$ Arc length of $f(x)$ over $[1,2] = \sqrt{17} - \frac{\sqrt{5}}{2} + \frac{1}{4}\left[\ln(\sqrt{17}+4) - \ln(\sqrt{5}+2)\right]$

(ii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \Rightarrow \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0$

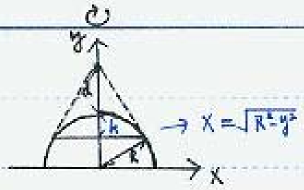


$\Rightarrow y' = -\frac{y^{1/3}}{x^{1/3}}$

$\Rightarrow \sqrt{1+(y')^2} = \frac{1}{x^{1/3}}$

$\Rightarrow S = \int_0^1 \frac{1}{x^{1/3}}\, dx = \frac{3}{2}$

$\therefore$ the length of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ is $4\times\frac{3}{2} = 6$.



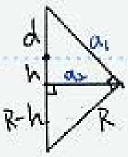
2. Let h be the height of the cap.

$\because$ the equation of right half of the circle is $X=\sqrt{R^2-y^2}$

$\Rightarrow X' = \frac{1}{2}(R^2-y^2)^{-\frac{1}{2}}\cdot(-2y) = \frac{-y}{\sqrt{R^2-y^2}}$

$\therefore$ surface area $= 2\pi\int_0^R \sqrt{R^2-y^2}\left(\frac{R}{\sqrt{R^2-y^2}}\right)dy - 2\pi\int_0^{R-h}\sqrt{R^2-y^2}\left(\frac{R}{\sqrt{R^2-y^2}}\right)dy$

$= 2\pi R h$



$a_1^2 = (d+h)^2 + a_2^2$; $a_2^2 = R^2-(R-h)^2$

$(d+R)^2 = R^2 + a_1^2 = R^2+(d+h)^2+a_2^2 = R^2+(d+h)^2+R^2-(R-h)^2$

$\Rightarrow h = \frac{dR}{d+R}$

Hence, surface area $= 2\pi R\cdot\left(\frac{dR}{d+R}\right)$

3.

(a) $f(X) = \ln X \Rightarrow f^{(K+1)}(X) = (-1)^K\frac{K!}{X^{K+1}}$

$\because |f^{(K+1)}(X)| = K!\cdot|X|^{-K-1}$ is decreasing for $X>0$.

$\therefore$ the maximum of $|f^{(K+1)}(X)|$ on $[1,c]$ is $|f^{(K+1)}(1)|$, for $c>1$.

(b) Error bound, $|T_n(c)-\ln c| \le K\cdot\frac{|c-1|^{n+1}}{(n+1)!}$

where $|f^{(n+1)}(X)|\le K$, for X between 1 and c

by (a), we may take $K = |f^{(n+1)}(1)| = n!$

Hence, $|T_n(c)-\ln c| \le \frac{|c-1|^{n+1}}{n+1}$

(c) By (b), $|T_n(1.5)-\ln 1.5| \le \frac{|1.5-1|^{n+1}}{n+1} = \frac{(0.5)^{n+1}}{n+1}$

$n=3$, $\frac{(0.5)^4}{4} = 0.015625 > 10^{-2}$

$n=4$, $\frac{(0.5)^5}{5} = 0.00625 < 10^{-2}$

Hence, $n=4$ will guarantee the desired accuracy.

4.

(a) 參考 §10.1 Example 2.

(b) $\frac{dx}{dt} = t \cdot \tan x$; $x(0) = 1$

$\cot x \cdot dx = t \cdot dt \Rightarrow \int \cot x \, dx = \int t \, dt$

$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{\sin x} d(\sin x) = \ln|\sin x| + C$

$\Rightarrow \ln|\sin x| = \frac{1}{2} t^2 + C$

$\Rightarrow |\sin x| = e^c \cdot e^{\frac{1}{2}t^2}$

$\Rightarrow \sin x = \pm e^c \cdot e^{\frac{1}{2}t^2}$

$\therefore x = \sin^{-1}(C_1 \cdot e^{\frac{1}{2}t^2})$, where $C_1 = \pm e^c$ is constant.

$x(0) = \sin^{-1} C_1 = 1 \Rightarrow C_1 = \sin 1$

Hence, $x = \sin^{-1}(\sin 1 \cdot e^{\frac{1}{2}t^2})$

(c) $y^2 \frac{dy}{dx} = x^{-3}$; $y(2) = 0$

$y^2 \cdot dy = x^{-3} \cdot dx \Rightarrow \int y^2 dy = \int x^{-3} dx$

$\Rightarrow \frac{1}{3} y^3 = -\frac{1}{2} x^{-2} + C$

$\Rightarrow y = (C_1 - \frac{3}{2} x^{-2})^{\frac{1}{3}}$, where $C_1 = 3C$ is constant.

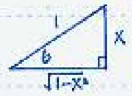
$y(2) = (C_1 - \frac{3}{8})^{\frac{1}{3}} = 0 \Rightarrow C_1 = \frac{3}{8}$

Hence, $y = (\frac{3}{8} - \frac{3}{2} x^{-2})^{\frac{1}{3}}$

(d) $\sqrt{1-x^2}\, y' = y^2$; $y(0) = 1$

$\frac{1}{y^2} dy = \frac{1}{\sqrt{1-x^2}} dx \Rightarrow \int \frac{1}{y^2} dy = \int \frac{1}{\sqrt{1-x^2}} dx$

$\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{1}{\cos\theta} \cos\theta \, d\theta = \theta + C = \sin^{-1} x + C$



$x = \sin\theta$
$\Rightarrow dx = \cos\theta \cdot d\theta$

$\Rightarrow y = -\frac{1}{\sin^{-1} x + C}$

$y(0) = -\frac{1}{\sin^{-1} 0 + C} = 1 \Rightarrow C = -1$

Hence, $y = \frac{1}{1 - \sin^{-1} x}$

5.

(a) $\frac{dy}{dt} = \frac{B \cdot v(y)}{A(y)}$, $v(y) = -8\sqrt{y}$ ft/s

1°, $\frac{r}{y} = \frac{4}{12} \Rightarrow r = \frac{1}{3}y \Rightarrow A(y) = \frac{\pi}{9}y^2$

$B = 4\pi\,\text{in}^2 = \frac{\pi}{36}\,\text{ft}^2$

$\Rightarrow \frac{dy}{dt} = -\frac{2}{\pi}y^{-\frac{3}{2}}$

2°, $\left(\frac{x}{2}\right)^2 + (y-10)^2 = 10^2 \Rightarrow x = 2\sqrt{20y-y^2} \Rightarrow A(y) = 40x = 80\sqrt{20y-y^2}$

$B = 9\pi\,\text{in}^2 = \frac{\pi}{16}\,\text{ft}^2$

$\Rightarrow \frac{dy}{dt} = -\frac{1\,\pi}{160\sqrt{20-y}}$.

(b) 1°, $\frac{dy}{dt} = -\frac{2}{\pi}y^{-\frac{3}{2}} \Rightarrow \frac{2}{5}y^{\frac{5}{2}} = -\frac{2}{\pi}t + C$

$y(0) = 12 \Rightarrow C = \frac{2}{5}(12)^{5/2}$

$\therefore y(t) = \left(12^{\frac{5}{2}} - \frac{5t}{\pi}\right)^{2/5}$

$y = 0 \Rightarrow 12^{\frac{5}{2}} - \frac{5t}{\pi} = 0 \Rightarrow t = \frac{1}{5}\pi\cdot(12)^{5/2}$ (≈ 313 seconds.)

2°, $\frac{dy}{dt} = -\frac{\pi}{160\sqrt{20-y}} \Rightarrow -\frac{2}{3}(20-y)^{\frac{3}{2}} = -\frac{1}{160}t + C$

$y(0) = 10 \Rightarrow C = \cancel{-\frac{2}{3}10^{\frac{3}{2}}}$

$\therefore y(t) = 20 - \left(\frac{3\pi}{320}t + 10^{\frac{3}{2}}\right)^{\frac{2}{3}}$

$y = 0 \Rightarrow 20 - \left(\frac{3\pi}{320}t + 10^{\frac{3}{2}}\right)^{\frac{2}{3}} = 0 \Rightarrow t = \frac{320}{3\pi}\left(20^{\frac{3}{2}} - 10^{\frac{3}{2}}\right)$

6. 參考 §10.3

7.

(a) $y' - (\ln x)\,y = x^x$

the integrating factor : $\alpha(x) = e^{\int -\ln x\,dx} = \frac{e^x}{x^x}$

$x^{-x}e^x y' - (\ln x)\,x^{-x}e^x y = e^x \Rightarrow (x^{-x}e^x y)' = e^x$

$\Rightarrow x^{-x}e^x y = e^x + C$

Hence, $y(x) = x^x + C\cdot x^x e^{-x}$

②

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7.

(b) $y' + \frac{1}{x+1} y = x^{-2}$, $y(1) = 2$

the integrating factor : $\alpha(x) = e^{\int \frac{1}{x+1} dx} = x + 1$

$(x+1)y' + y = x^{-1} + x^{-2} \Rightarrow ((x+1)y)' = x^{-1} + x^{-2}$

$\Rightarrow (x+1)y = \ln x - \frac{1}{x} + c$

$\therefore y(x) = \frac{1}{x+1}(\ln x - \frac{1}{x} + c)$

$y(1) = \frac{1}{2}(c-1) = 2 \Rightarrow c = 5$

Hence, $y(x) = \frac{1}{x+1}(\ln x - \frac{1}{x} + 5)$

(c) $y' + (\sec x) y = \cos x$

the integrating factor : $\alpha(x) = e^{\int \sec x \, dx} = \sec x + \tan x$

$(\sec x + \tan x) y' + (\sec^2 x + \sec x \cdot \tan x) y = 1 + \sin x$

$\Rightarrow ((\sec x + \tan x) y)' = 1 + \sin x$

$\Rightarrow (\sec x + \tan x) y = x - \cos x + C$

Hence, $y(x) = \frac{x - \cos x + C}{\sec x + \tan x}$

(d) $(\sin x)y' - (\cos x)y = 1$, $y(\frac{\pi}{4}) = 0.$

the integrating factor : $\alpha(x) = e^{\int -\cot x\, dx} = \csc x.$

$(\csc x)y' - (\frac{\cos x}{\sin^2 x})y = \csc^2 x$

$\Rightarrow (\frac{1}{\sin x} y)' = \csc^2 x$ $\qquad \frac{d}{dx}\cot x = -\csc^2 x$

$\Rightarrow (\csc x)y = -\cot x + C$

$\therefore y(x) = -\cos x + C \cdot \sin x$

$y(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}} + C \cdot \frac{1}{\sqrt{2}} = 0 \Rightarrow C = 1$

Hence, $y(x) = -\cos x + \sin x.$

8.

(a) $\lim_{n\to\infty} n^{\frac{1}{n}}$

$= \lim_{n\to\infty} e^{\ln(n^{\frac{1}{n}})} = e^{\lim_{n\to\infty} \ln(n^{\frac{1}{n}})}$

$= e^{\lim_{n\to\infty} \frac{\ln n}{n}} = e^{\lim_{n\to\infty} \frac{1}{n}}$

$= 1$

(b) $\lim_{n\to\infty} (1+\frac{1}{n^2})^n$

$(1+\frac{1}{n^2})^n = e^{\ln[(1+\frac{1}{n^2})^n]} = e^{\frac{\ln(1+\frac{1}{n^2})}{\frac{1}{n}}}$

$\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n^2})}{1/n} = \lim_{n\to\infty} \frac{(-2n^{-3})\frac{1}{1+n^{-2}}}{-n^{-2}} = \lim_{n\to\infty} \frac{\frac{2}{n}}{1+\frac{1}{n^2}} = 0$

$\therefore \lim_{n\to\infty} (1+\frac{1}{n^2})^n$

$= \lim_{n\to\infty} e^{\ln[(1+\frac{1}{n^2})^n]}$

$= e^{\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n^2})}{1/n}}$

$= 1.$

(I) Calculus Test II 2008/08/31

25% 1. Determin the convergence or divergence of the following series respectively.
(5%×5)

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ (b) $\Sigma \frac{\ln n}{n^{1.0001}}$ (c) $\sum_{n=1}^{\infty} \frac{n^3}{n!}$

(d) $\sum_{n=0}^{\infty} (-1)^n \frac{n!}{100^n}$ (e) $\sum_{n=1}^{\infty} (-1)^n \cos \frac{1}{n}$

20% 2. Find the range and radius of convergence:
(5%×4)

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (x-5)^n$ (b) $\sum_{n=1}^{\infty} \frac{x^n}{(n!)^2}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$ (d) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

10% 3. Find a power series solution of the following Eq.
(5%×2)

(a) $\begin{cases} x^2 y'' + x y' + (x^2-1) y = 0 \\ y'(0) = 3 \end{cases}$

(b) $\begin{cases} y'' - x y' + y = 0 \\ y(0) = 1, \ y'(0) = 0 \end{cases}$

15% 4 Find the Taylor series of $f(x)$ at $x=a$.
(5%×3)

(a) $f(x) = e^{-x^2}, \ a=0$

(b) $f(x) = \ln(1+x), \ a=0$

(c) $f(x) = \sinh x, \ a=0$

5. (a) Let $F(x) = \int_0^x \frac{\sin t}{t}$.

20%

Show that

$$F(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$

Evaluate $F(1)$ to three decimal places.

(b) Let $J = \int_0^1 \sin(x^2)\, dx$

Express J as an infinite series, and determine J to within an error less than 10^{-2}.

6. (a) Find parametric equation for the cycloid generated by a point P on the circle with radius R_0.

10%
(5%×2)

(b) Calculate the length s of half arch of the cycloid generated by the circle with radius 4.

7. (a) Sketch the graph of $r = \sin 3\theta$ and compute the area of one "petal".

10%
(5%×2)

(b) Find the total area enclosed by the cardioid $r = 1 - \cos\theta$.

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1.

(a) Let $f(x) = \frac{1}{x \cdot \ln x}$, $x \geq 2$

$\Rightarrow f(x) > 0$ and $f(x)$ is decreasing for $x \geq 2$

$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x \cdot \ln x}\,dx$ let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$= \int_{x=2}^{x=\infty} \frac{1}{u}\,du$

$= \ln|\ln x| \Big|_{x=2}^{x=\infty} = \infty$

$\therefore \int_2^\infty f(x)\,dx$ diverges.

by integral test, we know $\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$ diverges.

Note: Ratio test is inconclusive for $\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$.

(b) For $n \geq 1$, $\ln n \leq n$

$\because \ln n = \frac{1}{k} \ln(n)^k \leq \frac{1}{k}(n)^k$

$\Rightarrow \frac{\ln n}{n^{1.0001}} \leq \frac{1}{k} \frac{n^k}{n^{1.0001}} = \frac{1}{k} \frac{1}{n^{1.0001-k}}$

$\sum \frac{1}{n^{1.0001-k}}$ converges, if $1.0001 - k > 1$ ($\Rightarrow k < 0.0001$)

take $k = 0.00005$, we have

$\frac{\ln n}{n^{1.0001}} \leq \frac{1}{0.00005} \cdot \frac{1}{n^{1.00005}}$ and $\sum \frac{1}{n^{1.00005}}$ converges.

by the comparison test, we know $\sum \frac{\ln n}{n^{1.0001}}$ converges.

(c) Let $a_n = \frac{n^3}{n!}$, $n \geq 1$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3}{(n+1)!}\frac{n!}{n^3}\right| = \frac{(n+1)^2}{n^3} \xrightarrow{n\to\infty} 0$

by the ratio test, we know $\underline{\sum_{n=1}^{\infty}\frac{n^3}{n!} \text{ converges.}}$

(d) Let $a_n = (-1)^n \frac{n!}{100^n}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{100} \xrightarrow{n\to\infty} \infty$

by the ratio test, we know $\underline{\sum_{n=0}^{\infty}(-1)^n\frac{n!}{100^n} \text{ diverges.}}$

(e) $\because \lim_{n\to\infty} \cos\frac{1}{n} = 1 \neq 0$

Hence, $\underline{\sum_{n=1}^{\infty}(-1)^n \cdot \cos\frac{1}{n} \text{ diverges.}}$

2.

(a) Let $a_n = \frac{(-1)^n}{n}$

$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1$

$\Rightarrow$ the radius of convergence is $R = \frac{1}{r} = 1$

$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n}(x-5)^n$ converges, $|x-5|<1$

$x = 4$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(x-5)^n = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

$x = 6$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(x-5)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges. $\longrightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is conditionally convergent.

Hence, the radius of convergence $= 1$.

$\underline{\text{the range of convergence : } (4,6]}$

(b) Let $a_n = \frac{1}{(n!)^2}$

$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{(n+1)^2} = 0$

$\Rightarrow$ the radius of convergence is $R = \infty$

the range of convergence is $\mathbb{R}$. i.e. $\sum_{n=1}^{\infty} \frac{x^n}{(n!)^2}$ conv. for $x \in \mathbb{R}$.

(c) Let $a_n = \frac{(-1)^n}{\sqrt{n^2+1}}$

$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \sqrt{\frac{n^2+1}{n^2+2n+2}} = 1.$

$\Rightarrow$ the radius of convergence is $R = 1$

$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}} x^n$ converges, $|x| < 1$

$x = 1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ converges.

$x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}} x^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges. $\rightarrow$ $\because \frac{1}{\sqrt{n^2+n^2}} \le \frac{1}{\sqrt{n^2+1}}$, $n \ge 1$ & $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2}\cdot n}$ div.

Hence, the radius of convergence $= 1$

the range of convergence : $(-1, 1]$

(d) Let $a_n = \frac{1}{n^2}$

$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$

$\Rightarrow$ the radius of convergence is $R = 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges, $|x| < 1$

$x = 1$, $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$x = -1$, $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges.

Hence, the radius of convergence $= 1$

the range of convergence : $[-1, 1]$

3.

(a) Let $P(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow P'(x) = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$; $P''(x) = \sum_{n=2}^{\infty} n(n-1) \cdot a_n x^{n-2}$

$$x^2 y'' + x y' + (x^2 - 1) y$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} \left(a_n x^{n+2} - a_n x^n \right)$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^n + a_1 x + \sum_{n=2}^{\infty} n a_n x^n + \cancel{a_0 x^2 + a_1 x^3 + \sum_{n=2}^{\infty} a_n x^{n+2}} \ \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$- a_0 - a_1 x - \sum_{n=2}^{\infty} a_n x^n$$

$$= - a_0 + \sum_{n=2}^{\infty} \left[n(n-1) a_n + n a_n + a_{n-2} - a_n \right] x^n$$

$$= 0$$

$\Rightarrow a_0 = 0$ & $n(n-1) a_n + n a_n + a_{n-2} - a_n = 0$, for $n \geq 2$

$\Rightarrow a_0 = 0$ & $a_n = - \frac{a_{n-2}}{n^2 - 1}$, for $n \geq 2$

$\therefore a_2 = -\frac{a_0}{3} = 0$, $a_4 = -\frac{a_2}{15} = 0$, $a_6 = -\frac{a_4}{35} = 0$ ...

$\because y'(0) = 3 \Rightarrow P'(0) = a_1 = 3$

$\left(\therefore a_3 = -\frac{a_1}{8} = -\frac{3}{8} \right.$, $a_5 = -\frac{a_3}{24} = +\frac{1}{64}$, $\left. a_7 = -\frac{a_5}{48} = -\frac{1}{3072} \ \ldots \right)$

$\left(a_3 = -\frac{3}{3^2-1} \right.$, $a_5 = +\frac{3}{(5^2-1)(3^2-1)}$, $\left. a_7 = -\frac{3}{(7^2-1)(5^2-1)(3^2-1)} \right)$

Let $n = 2k+1$, $k \geq 1$

$\Rightarrow a_{2k+1} = -\frac{a_{2k-1}}{4k(k+1)}$

$a_3 = -\frac{a_1}{4 \cdot 1 \cdot (1+1)}$, $a_5 = -\frac{a_3}{4 \cdot 2 \cdot (2+1)} = + \frac{a_1}{4^2 \cdot (2 \cdot 1) \cdot ((2+1) \cdot (1+1))}$,

$a_7 = -\frac{a_5}{4 \cdot 3 \cdot (3+1)} = - \frac{a_1}{4^3 \cdot (3 \cdot 2 \cdot 1) \cdot ((3+1)(2+1)(1+1))}$, ...

$\Rightarrow a_{2k+1} = (-1)^k \frac{3}{4^k \cdot k! \cdot (k+1)!}$

Hence, $\underline{P(x) = \sum_{k=0}^{\infty} (-1)^k \frac{3}{4^k \cdot k! \cdot (k+1)!}}$

國立中央大學考試試卷

National Central University Examination Answer Sheet

☑ 平時考(Quiz) ☐ 期中考(Midterm) ☐ 期末考(Final)			評 分 (Score)
科 目 (Course Title)	日 期 (Date) 8/31		
系/級 (Department / Grade)	班/組 (Class)		
學 號 (Student ID)	姓 名 (Name)		

(b) Let $P(x)=\sum_{n=0}^{\infty} a_n x^n$

$y''-xy'+y$

$=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=1}^{\infty} n a_n x^n+\sum_{n=0}^{\infty} a_n x^n$

$=\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n-\sum_{n=1}^{\infty} n a_n x^n+\sum_{n=0}^{\infty} a_n x^n$

$=2a_2+a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-na_n+a_n\right]x^n$

$=0$

$\rightarrow a_{n+2}=\frac{n-1}{(n+2)(n+1)}a_n$

$\Rightarrow 2a_2+a_0=0$ & $a_n=\frac{(n+2)(n+1)a_{n+2}}{n-1}$, for $n\geq 1$

$\because y(0)=1 \Rightarrow a_0=1$

$y'(0)=0 \Rightarrow a_1=0$

$\therefore a_2=-\frac{a_0}{2}=-\frac{1}{2}$

1° $a_3=0$, $a_5=\frac{2}{(3+2)(3+1)}a_3=0$, $a_7=\frac{4}{(5+2)(5+1)}a_5=0$, ...

2° let $n=2k \Rightarrow a_{2k+2}=\frac{2k-1}{(2k+2)(2k+1)}a_{2k}=\frac{2k-1}{2(k+1)(2k+1)}a_{2k}$.

$a_2=\frac{-1}{2\cdot1\cdot1}a_0$, $a_4=\frac{1}{2\cdot(1+1)\cdot(2+1)}a_2=\frac{-1}{4\cdot3\cdot2}a_0$

$a_6=\frac{3}{2(2+1)(4+1)}a_4=-\frac{3}{6\cdot5\cdot4\cdot3\cdot2}a_0$

$\Rightarrow a_{2k}$

2° $a_4=\frac{1}{4\cdot3}a_2=\frac{-1}{4\cdot3\cdot2}a_0$, $a_6=\frac{3}{6\cdot5}a_4=-\frac{+3}{6\cdot5\cdot4\cdot3\cdot2}a_0$

$a_8=\frac{5}{8\cdot7}a_6=-\frac{5\cdot3}{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}a_0$

$a_{10}=\frac{7}{10\cdot9}a_8=-\frac{7\cdot5\cdot3}{10!}a_0$

So $P(x)=1-\frac{1}{2}x^2-\frac{1}{4!}x^4-\frac{3}{6!}x^6-\ldots\ldots$

4.

(a) $\because e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \quad , \forall x$

$\therefore e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$

Hence, $\underline{e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}}$

(b) $\because \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + \dots \quad , |x| < 1$

$\Rightarrow \ln(1+x) = \int \frac{1}{1+x} dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

converges to $\ln(1+x)$ for $|x|<1$ and $x=1$

Hence, $\underline{\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}}$

(c) $\because \sinh x = \frac{e^x - e^{-x}}{2}$

$\Rightarrow \sinh x = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right)$

$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} (1 - (-1)^n)$

$\because 1 - (-1)^n = \begin{cases} 0, & n = \text{even} \\ 2, & n = \text{odd} \end{cases}$

Hence, $\underline{\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$

5.

(a) 1° $\frac{\sin t}{t} = \frac{1}{t} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$

$\Rightarrow F(x) = \int_0^x \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\cdot(2n+1)!}$

$= x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \dots$

2° $F(1) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)\cdot(2n+1)!}$

$\because F(1)$ is an alternating series with decreasing terms

$\Rightarrow \left| F(1) - \sum_{n=0}^{N} (-1)^n \frac{1}{(2n+1)\cdot(2n+1)!} \right| \le \frac{1}{(2N+3)\cdot(2N+3)!}$

$N=1, \quad \frac{1}{(2+3)\cdot(2+3)!} = \frac{1}{600} \approx 0.0017 = 1.7 \times 10^{-3}$

$N=2, \quad \frac{1}{(4+3)\cdot(4+3)!} = \frac{1}{35280} \approx 0.000028 = 2.8 \times 10^{-5}$

For $N=2$, $\underline{\sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)\cdot(2n+1)!} = 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} = 0.94611}$

(b) 1° $\because \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \Rightarrow \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$

$\therefore J = \int_0^1 \sin x^2 \, dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 x^{4n+2} \, dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{4n+3}\right) = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \cdots$

2° $\because J$ is an alternating series with decreasing terms

$\Rightarrow \left| J - \sum_{n=0}^{N-1} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{4n+3}\right) \right| \leq \frac{1}{(4N+3)\cdot(2N+1)!}$

$N=1$, $\frac{1}{(4+3)\cdot(2+1)!} \approx 0.024 = 2.4 \times 10^{-2}$

$N=2$, $\frac{1}{(8+3)\cdot(4+1)!} \approx 0.00076 = 7.6 \times 10^{-4} < 10^{-2}$

For $N=2$ we obtain

$J \approx \frac{1}{3} - \frac{1}{42} \approx 0.30952$

6.

(a) 參考 §12.1 Example 7.

$x(t) = R_0 t - R_0 \cdot \sin t$

$y(t) = R_0 - R_0 \cdot \cos t$

(b) $x(t) = 4t - 4\cdot\sin t \Rightarrow x'(t) = 4 - 4\cdot\cos t$

$y(t) = 4 - 4\cdot\cos t \Rightarrow y'(t) = 4\cdot\sin t$

$\therefore S = \int_0^{\pi} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$

$= \int_0^{\pi} \sqrt{(4-4\cdot\cos t)^2 + (4\cdot\sin t)^2} \, dt$

$= \int_0^{\pi} \sqrt{32(1-\cos t)} \, dt$

$= \int_0^{\pi} \sqrt{64\left(\frac{1-\cos t}{2}\right)} \, dt$

$= \int_0^{\pi} \sqrt{64\cdot\sin^2\frac{t}{2}} \, dt$

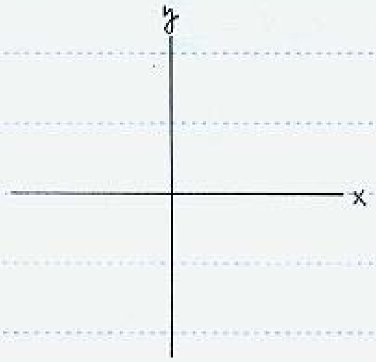
$= 8 \cdot \int_0^{\pi} \sin\frac{t}{2} \, dt$

$= -16\cdot\cos\frac{t}{2} \Big|_0^{\pi}$

$= -16\cdot(0-1)$

$= 16$

7.



(a) 參考 § 12.4 Example 2

$$\frac{1}{2}\int_0^{\pi/3} (\sin 3\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3} \frac{1-\cos 6\theta}{2} \, d\theta$$

$$= \underline{\frac{\pi}{12}}$$ ..... the area of one petal.

(b) $A = 2 \cdot \frac{1}{2} \int_0^{\pi} r^2 \, d\theta$

$$= \int_0^{\pi} (1-\cos\theta)^2 \, d\theta$$

$$= \int_0^{\pi} 1 - 2\cdot\cos\theta + \cos^2\theta \, d\theta$$

$$= \underline{\frac{3}{2}\pi}.$$