Calculus : Final Exam. 2009/09/09 (P1)

物 10% 他 10%

1 (i) If $f'(x)$ exists and is continuous on $[a, b]$, then Prove 數學 物理

$$\text{Arc length over } [a,b] = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

(ii) Using above (i) to find the arc length of $f(x)=x^2$ over $[0, 1]$ $\frac{\sqrt5}{2}+\frac{1}{4}\ln|2+\sqrt5|$

2 (i) Let $f^{(n+1)}(x)$ exists and be continuous $\forall x$ Find the nth Taylor polynomial

10% $T_n(x)$ for f centered at $x=a$, and Prove 數學 物理

10%

$$R_n(x) = f(x) - T_n(x) = \frac{1}{n!}\int_a^x (x-u)^n f^{(n+1)}(u)\,du$$

(ii) Let $\begin{cases} f(x) = \sin x \\ a=0 \end{cases}$. Show that the Taylor polynomial for f are

$$T_{2n-1}(x) = T_{2n}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

Use the Error bound with $n=4$ to show

$$\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \le \frac{|x|^5}{120} \quad \forall x$$

3 Whether the infinite series is convergent or divergent? why?

20% (i) $\sum_{n=2}^{\infty} e^{3-2n}$ (ii) $\sum_{n=1}^{\infty} (-1)^n n e^{-n}$

2% (iii) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (iv) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

(iv) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

4 Solving the following initial Value Problem:

10% (a) $\begin{cases} y'(t) = -3ty \\ y(0) = 6 \end{cases}$ (b) $\begin{cases} \sqrt{2-x^2}\,\frac{dy}{dx} = y^3 \\ y(0) = 3 \end{cases}$

10%

$y = 6e^{-\frac{3}{2}t^2}$ $y = \pm\sqrt{\frac{1}{-2\sin^{-1}\left(\frac{x}{\sqrt2}\right)+\frac19}}$

5 (a) Find parametric equation for the Cycloid generated by a point P on

10% the circle with radius R $x = Rt - R\sin t$ $y = R - R\cos t$

10% (b) Calculate the length s of one arch of the cycloid generated by the unit circle $(R=1)$ 8

6. Let $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$ 數學 物理

10%

Prove
$$f_{xx} + f_{yy} = f_{rr} + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r} f_r$$

P. 828 ex: 44

(hint: $x = g(t, s)$ $y = h(t, s)$

$f_{tt} = f_{xx}\left(\frac{\partial x}{\partial t}\right)^2 + 2 f_{xy}\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial t}\right) + f_{yy}\left(\frac{\partial y}{\partial t}\right)^2 + f_x \frac{\partial^2 x}{\partial t^2} + f_y \frac{\partial^2 y}{\partial t^2}$

7. Find the minimum and maximum value of the function subject to given constraint.

20%
20%

(a) $f(x, y) = x^2 y + x + y$, $xy = 4$ P. 848. ex 8

$y = \frac{4}{x}$ $f(x,y) = x^2 \frac{4}{x} + x + \frac{4}{x}$
$= 4x + x + \frac{4}{x}$
$= 5x + \frac{4}{x}$

沒有極大 極小,

$x \to 0^+$ $f(x, y) \to \infty$
$x \to 0^-$ $f(x, y) \to -\infty$

(b) $f(x, y) = x^2 y^4$, $x^2 + 2y^2 = 6$ P. 848. ex 10

極小 0
極大 8

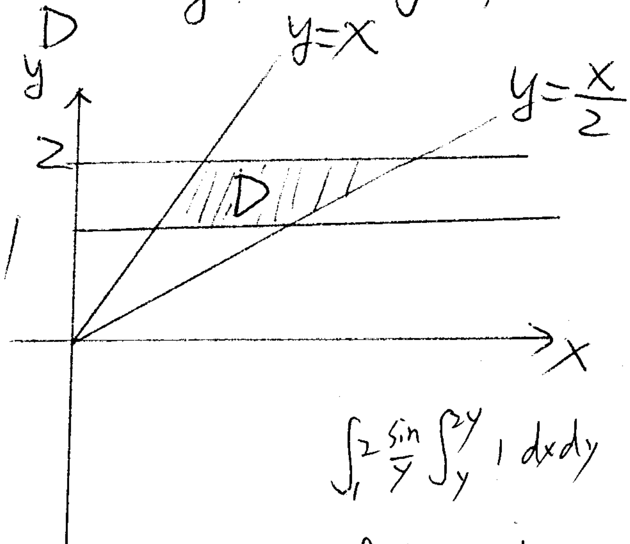
8. Evaluate the following integrals:

10%
20%

(i) $\iint_D e^{x+y}\, dx\, dy$, $D = 0 \le x, y \le 2$ P. 878 ex 46

$\int_0^2 \int_0^2 e^{x+y}\, dx\, dy$
$= \int_0^2 e^y (e^2 - 1)\, dy$
$= (e^2 - 1)^2$

(ii) $\iint_D \frac{\sin y}{y}\, dx\, dy$, P. 879 ex 51



$\int_1^2 \frac{\sin}{y} \int_y^{2y} 1\, dx\, dy$

$= \int_1^2 \frac{\sin y}{y}\, y\, dy$

$= \int_1^2 \sin y\, dy = -\cos y \Big|_1^2 = \cos 1 - \cos 2$