(P1) Calculus : Mid-T. Exam. 2009/08/24

1. (i) Let $f^{(n+1)}(x)$ exists and continuous $\forall x$.

10% (配4分) (a) Find $T_n(x)$: the nth Taylor polynomial of f centered at $x=a$

(b) Prove

$$R_n(x) = f(x) - T_n(x) = \frac{1}{n!}\int_a^x (x-u)^n f^{(n+1)}(u)\,du$$

P.495

(ii) Let $f(x) = \sin x$, $a=0$.

P.500 ex 25 Show that

(a) $T_{2n-1}(x) = T_{2n}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$

(b) $\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \le \frac{|x|^5}{120} \quad \forall x$

$|T_n(x) - f(x)| \le K\frac{|x-a|^{n+1}}{(n+1)!}$ $K=1$

2 (i) If $f'(x)$ exists and is continuous on $[a, b]$, then

10%

Prove : Arc length of f over $[a,b] = \int_a^b \sqrt{1+(f'(x))^2}\,dx$

(ii) Calculate the length of the astroid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

P.491 ex 15

3 Solving the following initial value Problems:

20%

(a) $\begin{cases} y'(t) = -3ty \\ y(0) = 5 \end{cases}$ P.507 example

(b) $\begin{cases} \sqrt{1-x^2}\,\frac{dy}{dt} = y^2 \\ y(0) = 1 \end{cases}$ P.512 ex.39 $y = \frac{1}{-\sin^{-1}x + C}$

(c) $\begin{cases} 2\frac{dy}{dx} + 6y + 4 = 0 \\ y(0) = 1 \end{cases}$ P.511 ex.17

(d) $\begin{cases} y'' + 2y' - 8y = 0 \\ y(0) = 1 \end{cases}$ P.512 ex.43

$= \pm e^{c}e^{-x} - \frac{2}{3}$

$= Ae^{-x} - \frac{2}{3}$

4. 15% A cylindrical tank of heigh 9 ft and radius 2 ft is filled with water. Water drains through a square hole of side 2 inch in the bottom. Determine the water level $y(t)$ at time t (seconds). How long does it take for the tank to go from full to empty? P.509 example 3

5. 16% Find the following limit respectively

(a) $\lim_{n\to\infty} n^{\frac{1}{n}}$ P.544 ex33 $=1$ (b) $\lim_{n\to\infty} \left(1+\frac{3}{n^2}\right)^n$

(c) $\lim_{n\to\infty} \frac{n!}{2^n}$ ∞ P.544 ex60 (d) $\lim_{n\to\infty} 3^{\frac{1}{n}}$ $=1$

6. 16% Whether the infinite series is convergent or divergent? Why?

P.558 example 1

(i) $\sum_{n=1}^{\infty} \frac{1}{n}$ P.555 ex38 divergent (ii) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ convergent

(iii) $\sum_{n=2}^{\infty} e^{3-2n}$ P.554 ex.35 converges (iv) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\, 3^n}$ P.559 example 4 converges

(v) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ P.560 example 6 converges (vi) $\sum_{n=1}^{\infty} (-1)^n n e^{-n}$ P.590 ex 25 converges

(vii) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ P.566 example 2 diverges (viii) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ P.595 ex 18 converges Ratio test

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物) 7. 15% Show that if $a_n \geq 0$ and $\lim_{n\to\infty} n^2 a_n$ exists, then $\sum_{n=1}^{\infty} a_n$ converges