(P1) Calculus Test (Mid-T. Exam) 2009/08/24

1. Find the solution $y(t)$:

(i) (a) $y' - (\ln x)\, y = x^x$

(b) $3y' + \frac{5}{x+1}\, y = 2x^{-2}, \quad y(1) = 2$

(c) $(\sin 3x)\, y' - (\cos 3x)\, y = 5, \quad y(\frac{\pi}{4}) = 0$

(d) $y'' + 2y' - 8y = 0, \quad y(0) = 2$

(e) $2\frac{dy}{dx} + 6y + 4 = 0, \quad y(0) = 3$

2. Find the nth Taylor polynomial of f at a.

(a) $f(x) = \sin x, \quad a = 0$

(b) $f(x) = e^{2x}, \quad a = 0$

(c) $f(x) = \frac{1}{1+x}, \quad a = 1$ (d) $f(x) = \ln x, \quad a = 1$.

3. Find the following limit

(a) $\lim_{n\to\infty} \left(1 + \frac{5}{n^2}\right)^n$ (b) $\lim_{n\to\infty} \frac{n!}{3^n}$

(c) $\lim_{n\to\infty} n^{\frac{2}{n}}$ (d) $\lim_{n\to\infty} \frac{\ln n}{n^{0.01}}$

4. Find the arc length of $f(x)$ over $[a,b]$:

(i) $f(x) = 2x^2, \quad [a,b] = [0,1]$

(ii) the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$

(iii) $f(x) = \left(\frac{x}{2}\right)^4 + \frac{1}{2x^2}, \quad [a,b] = [2,4]$

(iv) $f(x) = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}}, \quad [a,b] = [2,8]$

5. Whether the infinite series is convergent or divergent ? why ?

(i) $\sum_{n=1}^{\infty} \frac{1}{n}$ (ii) $\sum_{n=2}^{\infty} e^{3-2n}$

(iii) $\sum_{n=1}^{\infty} n e^{-n^2}$ (iv) $\sum_{n=1}^{\infty} \left(1-\cos\frac{1}{n}\right)$

(v) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ (vi) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

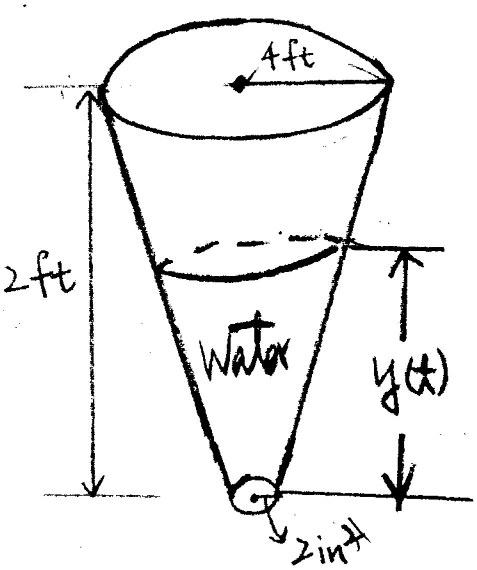
(vii) $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$ (viii) $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}\cdot \ln n}$

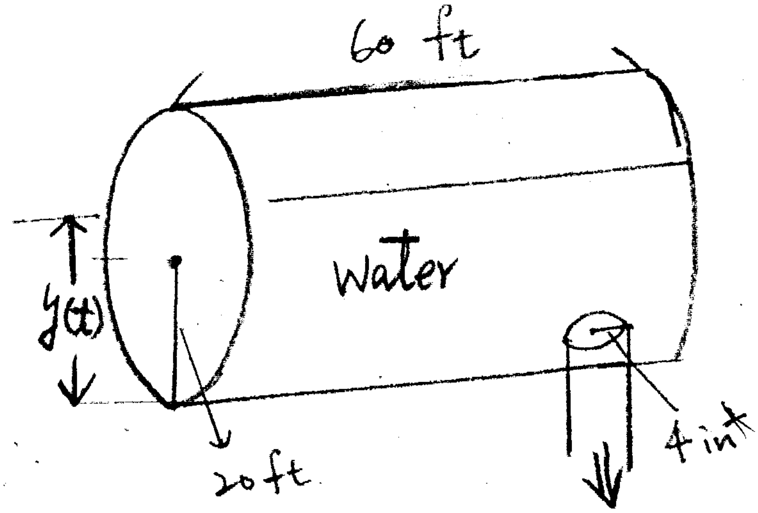
(ix) $\sum_{n=3}^{\infty} \frac{n^3}{\sqrt{n^4-2n^2+1}}$ (x) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$

6. A tank, in the following respectively, filled with water. Let $y(t)$ be the water level at time t.

(a) Find the differential equation satisfied by $y(t)$ and solve the $y(t)$.

(b) How long does it take for the tank to empty ?





國立中央大學考試試卷
National Central University Examination Answer Sheet

☐ 平時考(Quiz) ☑ 期中考(Midterm) ☐ 期末考(Final)			評分 (Score)
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1.

(a) the integrating factor : $\alpha(x) = e^{\int -\ln x \, dx} = \frac{e^x}{x^x}$

原式 $\Rightarrow (x^{-x} e^x y)' = e^x$

$\therefore x^{-x} e^x y = e^x + c$

Hence, $\underline{y(x) = x^x + c x^x e^{-x}}$

(b) the integrating factor : $\alpha(x) = e^{\int \frac{1}{x+1} dx} = x+1$

原式 $\Rightarrow ((x+1) y)' = x^{-1} + x^{-2}$

$\therefore (x+1) y = \ln x - \frac{1}{x} + c$

Hence, $\underline{y(x) = \frac{1}{x+1}(\ln x - \frac{1}{x} + c)} = \underline{\frac{1}{x+1}(\ln x - \frac{1}{x} + 5)}$

$\because y(1) = 2$

$\Rightarrow 2 \cdot 2 = \ln 1 - 1 + c$

$\Rightarrow c = 5$

(c) 原式 $\Rightarrow y' - \cot(3x) \cdot y = 5 \cdot \csc 3x$

the integrating factor : $\alpha(x) = e^{\int -\cot 3x \, dx} = \underline{\sin^{-\frac{1}{3}} 3x}$.

$((\sin 3x)^{-1/3} y)' = 5 (\sin 3x)^{-4/3}$

(d) Let $y = e^{ax}$

$\Rightarrow a^2e^{ax} + 2ae^{ax} - 8e^{ax} = e^{ax}(a^2+2a-8) = 0$

$\Rightarrow a^2 + 2a - 8 = 0$

$\Rightarrow a = -4 \ , \ 2$

$\therefore y(x) = c_1 e^{-4x} + c_2 e^{2x}$

$\because \left.\begin{array}{l} y(0) = c_1 + c_2 = 2 \\ y'(0) = -4c_1 + 2c_2 = 2 \end{array}\right\} \Rightarrow c_1 = \frac{1}{3} \ , \ c_2 = \frac{5}{3}$

Hence, $\underline{y(x) = \frac{1}{3}e^{-4x} + \frac{5}{3}e^{2x}}$

(e) 原式 $\Rightarrow \frac{dy}{3y+2} = -dx$

$\Rightarrow \ln|3y+2| = -x + C$

$\because y(0) = 3 \Rightarrow C = \ln 2$

$\Rightarrow \ln|3y+2| = -x + \ln 2$

Hence, $\underline{y(x) = \pm\frac{2}{3}e^{-x} - \frac{2}{3}}$

2.

(a) $f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}(0) = -1$

$\vdots$

$\Rightarrow T_0(x) = 0 \ , \quad T_3(x) = T_4(x) = x - \frac{1}{3!}x^3$

$T_1(x) = T_2(x) = x \ , \quad T_5(x) = T_6(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

Hence, in general,

$\underline{T_{2n+1}(x) = T_{2n+2}(x) = x - \frac{1}{3!}x^3 + \dots + (-1)^n\frac{1}{(2n+1)!}x^{(2n+1)}}$

(b) $f(x) = e^{2x} \Rightarrow f(0) = 1$

$f'(x) = 2e^{2x} \Rightarrow f'(0) = 2$

$f''(x) = 4e^{2x} \Rightarrow f''(0) = 4$

$\vdots$

$\Rightarrow T_0(x) = 1$, $T_1(x) = 1 + 2x$, $T_2(x) = 1 + 2x + \frac{4}{2!}x^2$

$T_3(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3$, ...

Hence, in general,

$\underline{T_n(x) = 1 + 2x + \frac{2^2}{2!}x^2 + \dots + \frac{2^n}{n!}x^n}$

(c) $f(x) = \frac{1}{1+x} \Rightarrow f(1) = \frac{1}{2}$

$f'(x) = \frac{-1}{(1+x)^2} \Rightarrow f'(1) = \frac{-1}{2^2}$

$f''(x) = \frac{2}{(1+x)^3} \Rightarrow f''(1) = \frac{2}{2^3}$

$f^{(3)}(x) = \frac{-2\cdot 3}{(1+x)^4} \Rightarrow f^{(3)}(x) = \frac{-6}{2^4}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^n \cdot n!}{(x+1)^{n+1}}$

$f^{(n)}(1) = \frac{(-1)^n \cdot n!}{2^{n+1}}$

$\Rightarrow T_0(x) = \frac{1}{2}$, $T_1(x) = \frac{1}{2} - \frac{1}{4}(x-1)$, $T_2(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2$

$T_3(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3$, ...

Hence, in general

$\underline{T_n(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 + \dots + (-1)^n \frac{1}{2^{n+1}}(x-1)^n}$

(d) $f(x) = \ln x \Rightarrow f(1) = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$

$\vdots$

$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}$, $n \geq 1$

$f^{(n)}(1) = (-1)^{n+1}\cdot(n-1)!$

$\Rightarrow T_0(x) = 0$, $T_1(x) = (x-1)$, $T_2(x) = (x-1) - \frac{1}{2}(x-1)^2$, ...

Hence, in general,

$\underline{T_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots + (-1)^{n+1}\frac{(x-1)^n}{n}}$

3.

(a) $\lim_{n\to\infty}\left(1+\frac{5}{n^2}\right)^n = e^{\lim_{n\to\infty} \ln\left(1+\frac{5}{n^2}\right)^n}$

$= e^{\lim_{n\to\infty} \ln\left(1+\frac{5}{n^2}\right)/\frac{1}{n}}$

$= e$

(b) $\frac{n!}{3^n} = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{3\cdot 3\cdot 3\cdot \ldots \cdot 3} = \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{3}{3}\cdot\left(\frac{4}{3}\cdot\frac{5}{3}\cdot\ldots\cdot\frac{n-1}{3}\right)\frac{n}{3}$

$= \left(\frac{4}{3}\cdot\frac{5}{3}\cdot\ldots\cdot\frac{n-1}{3}\right)\cdot\frac{2n}{27}$

$\Rightarrow \frac{n!}{3^n} > \frac{2n}{27}$

$\because \lim_{n\to\infty}\frac{2n}{27} = \infty$

Hence, $\lim_{n\to\infty}\frac{n!}{3^n} = \infty$

(c) $\lim_{n\to\infty} n^{\frac{2}{n}} = e^{\lim_{n\to\infty}\frac{\ln n}{/2}}$

$= e.$

(d) $\lim_{n\to\infty}\frac{\ln n}{n^{0.01}} = \lim_{n\to\infty}\frac{1}{0.01} n^{-0.01} = 0$

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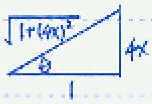
4.

(i) Arc length of $f(x) = 2x^2$ on $[0,1]$

$$= \int_0^1 \sqrt{1+(4x)^2}\, dx \qquad \text{let } \tan\theta = 4x$$



$$\Rightarrow \frac{1}{4}\sec^2\theta \cdot d\theta = dx$$

$$\sqrt{1+(4x)^2} = \sec\theta$$

$$= \frac{1}{4}\int_{x=0}^{x=1} \sec^3\theta \, d\theta$$

$$\because \int \sec^3\theta \, d\theta$$

$$= \frac{1}{2}\sec\theta \cdot \tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$\Rightarrow \frac{1}{4}\int_{x=0}^{x=1} \sec^3\theta \, d\theta$$

$$= \frac{1}{4}\left[2x\cdot\sqrt{1+(4x)^2}\,\Big|_{x=0}^{1} + \frac{1}{2}\ln\left|\sqrt{1+(4x)^2}+4x\right|\Big|_{x=0}^{1}\right]$$

$$= \frac{1}{4}\left[2\sqrt{17} + \frac{1}{2}\ln(\sqrt{17}+4)\right]$$

Hence, arc length $= \frac{1}{2}\sqrt{17} + \frac{1}{8}\ln(\sqrt{17}+4)$.

(iii) Arc length of $f(x) = \left(\frac{x}{2}\right)^4 + \frac{1}{2x^2}$ on $[2,4]$

$$= \int_2^4 \sqrt{\left(\frac{x^3}{4}+\frac{1}{x^3}\right)^2}\, dx$$

$$= \int_2^4 \frac{x^3}{4} + \frac{1}{x^3}\, dx$$

$$= \frac{1}{16}x^4 - \frac{1}{2x^2}\Big|_{x=2}^{4}$$

$$= \frac{483}{32}$$

(iv) Arc length of $f(x) = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}}$ on $[2, 8]$

$$= \int_2^8 \sqrt{\left(\frac{1}{2}\sqrt{x} + \frac{1}{2\sqrt{x}}\right)^2}\, dx$$

$$= \int_2^8 \frac{\sqrt{x}}{2} + \frac{1}{2\sqrt{x}}\, dx$$

$$= \frac{17\sqrt{2}}{3}$$

5.

(i) 參考 § 11.3 Example 1.

(ii) $\sum_{n=2}^{\infty} e^{3-2n} = \sum_{n=2}^{\infty} e^3 \cdot \left(\frac{1}{e^2}\right)^n$

$\because \frac{1}{e^2} < 1$

$\therefore \sum_{n=2}^{\infty} e^{3-2n} = \frac{e}{e^2-1}$ converges.

(iii) let $f(x) = xe^{-x^2}$, $x \geq 1$

$\Rightarrow f'(x) = (1-2x^2)e^{-x^2} < 0$, for $x \geq 1$

$\Rightarrow f(x)$ is decreasing for $x \geq 1$

$\int_2^{\infty} xe^{-x^2}dx = \lim_{a\to\infty}\int_2^a xe^{-x^2}dx = -\frac{1}{2}\lim_{a\to\infty}(e^{-a^2} - e^{-4}) = \frac{1}{2}e^{-4}$

$\therefore \sum_{n=1}^{\infty} ne^{-n^2}$ converges.

(iv) $\because \lim_{n\to\infty} \frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{-\frac{1}{n^2}\sin\frac{1}{n}}{-\frac{2}{n^3}} = \frac{1}{2} > 0$ ※ $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges

by the limit comparison test we know

$\sum_{n=1}^{\infty}\left(1-\cos\frac{1}{n}\right)$ converges.

(v) $\because \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 > 0$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

by the limit comparison test we know

$\sum_{n=1}^{\infty} \sin\frac{1}{n}$ diverges.

(vi) $\because \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$ & $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

$\therefore \sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ converges absolutely.

$\Rightarrow \sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ converges. ※ abso. conv. $\Rightarrow$ conv.

(vii) $\because \lim_{n\to\infty} \frac{n^4}{n^3+1} = \infty$.

by divergence test we know

$\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$ diverges.

(viii) let $f(x) = \frac{1}{x^{1/2}\cdot \ln x}$, $x \ge 2$

$\Rightarrow f'(x) = \frac{-x^{-1/2}(\frac{1}{2}\ln x + 1)}{x\cdot(\ln x)^2} < 0$, for $x \ge 2$

$\Rightarrow f(x)$ is decreasing for $x \ge 2$

$\int_2^{\infty} \frac{1}{x^{1/2}\cdot\ln x}\,dx = \lim_{a\to\infty}\int_2^a \frac{1}{x^{1/2}\cdot \ln x}\,dx$

$\because \frac{1}{x^{1/2}} > \frac{1}{x}$ for $x \ge 1$

$\Rightarrow \frac{1}{x^{1/2}\cdot\ln x} > \frac{1}{x\cdot \ln x} > 0$ for $x \ge 1$ &

$\int_2^{\infty}\frac{1}{x\cdot\ln x}\,dx = \int_{x=2}^{x\to\infty}\frac{1}{u}\,du$, $u = \ln x$

$= \ln(\ln x)\Big|_{x=2}^{\infty}$

$= \infty$

$\therefore \int_2^{\infty}\frac{1}{x^{1/2}\ln x}\,dx$ diverges.

by integral test we know

$\sum_{n=2}^{\infty}\frac{1}{n^{1/2}\cdot \ln n}$ diverges.

(ix) $\frac{n^3}{\sqrt{n^4-2n^2+1}} = \frac{n^3}{n^2-1}$, for $n \geq 3$

$\because \lim_{n\to\infty} \frac{\frac{n^3}{n^2-1}}{n} = 1 > 0$ & $\sum_{n=3}^{\infty} n$ diverges

by the limit comparison test we know

$\sum_{n=3}^{\infty} \frac{n^3}{\sqrt{n^4-2n^2+1}}$ diverges.

(x) let $a_n = \frac{1}{(2n+1)!}$

$\because \{a_n\}$ is a decreasing sequence and $\lim_{n\to\infty} a_n = 0$

by the Leibniz test we know

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$ converges.

6. ※ $\frac{dy}{dt} = -\frac{8B\sqrt{y}}{A(y)}$

(a) 1° $\frac{r}{y} = \frac{4}{12} \Rightarrow r = \frac{y}{3} \Rightarrow A(y) = \frac{\pi}{9} y^2$

$B = 4\pi \ in^2 = \frac{\pi}{36} \ ft^2$

$\therefore \frac{dy}{dt} = -2y^{-\frac{3}{2}}$

2° $\left(\frac{x}{2}\right)^2 + (y-20)^2 = 400 \Rightarrow x = 2\sqrt{40y-y^2} \Rightarrow A(y) = 120\sqrt{40y-y^2}$

$B = 16\pi \ in^2 = \frac{\pi}{9} \ ft^2$

$\therefore \frac{dy}{dt} = -\frac{\pi}{135\sqrt{40-y}}$

(b) 1° $\frac{dy}{dt} = -2y^{-\frac{3}{2}} \Rightarrow \frac{2}{5} y^{\frac{5}{2}} = -2t + C$

$\because y(0) = 12 \Rightarrow C = \frac{2}{5}(12)^{\frac{5}{2}}$

$\Rightarrow y(t) = \left((12)^{5/2} - 5t\right)^{2/5}$

$y(t) = 0 \Rightarrow t = \frac{1}{5}(12)^{5/2}$

2° $\frac{dy}{dt} = -\frac{\pi}{135\sqrt{40-y}} \Rightarrow -\frac{2}{3}(40-y)^{\frac{3}{2}} = -\frac{\pi}{135} t + C$

$\because y(0) = 40 \Rightarrow C = 0$

$\Rightarrow y(t) = 40 - \left(\frac{\pi t}{90}\right)^{2/3}$

$y(t) = 0 \Rightarrow t = \frac{90}{\pi}(40)^{3/2}$