微積分聯合教學　段考四　Mar 30, 2010

考試時間 120 分鐘，題目卷為兩張紙，共三頁，滿分 120 分。所有題目的答案都請依題號順序依序寫在答案卷上，而是非與填充題必須寫在第一頁。答案卷務必寫學號、姓名，題目卷不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘前不得離場。考試期間禁止使用字典、計算機及任何通訊器材，監試人員不得回答任何關於試題的疑問。

是非題 (*20* points)，請答 **T** (True) 或 **F** (False)。每題 2 分。(請依題號順序依序寫在答案卷第一頁上)

1. Suppose $a_n \geq 0$ for all n. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} {a_n}^3$ also converges.

T　Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$. Therefore there exists a positive integer N such that $|a_n| < 1$ for all $n \geq N$.
So we have $0 \leq a_n^3 \leq a_n \cdot 1 \cdot 1 = a_n$ for all $n \geq N$.
Thus $\sum_{n=1}^{\infty} a_n^3$ also converges by comparison test.

2. If a sequence is bounded from above and below, then the sequence converges.

F　Counterexample:
The sequence $\{(-1)^n\}_{n\geq 0} = 1, -1, 1, -1, 1, \ldots$ is bounded above by 1 and below by -1, but the sequence does not converge.

3. The series $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{(n^2+1)^{\frac{1}{3}}}$ converges absolutely.

F　$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \dfrac{1}{(n^2+1)^{\frac{1}{3}}} \right| = \sum_{n=1}^{\infty} \dfrac{1}{(n^2+1)^{\frac{1}{3}}}$. Observe that $\lim_{n\to\infty} \dfrac{\frac{1}{(n^2+1)^{1/3}}}{\frac{1}{(n^2)^{1/3}}} = 1$.
Since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges by p-series test,
$\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^{1/3}}$ also diverges by limit comparison test.

4. The sequence $\{a_n\} = \{\sqrt{n+1} - \sqrt{n}\}$ is decreasing and bounded below. (§11.1 Ex10)

T　Let $f(x) = \sqrt{x+1} - \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0$ for $x > 0$
$\Rightarrow f(x)$ is decreasing for $x > 0$
$\Rightarrow a_n = f(n)$ is also decreasing.
$a_n = \sqrt{n+1} - \sqrt{n} > 0, \ \forall n$
$\Rightarrow \{a_n\}$ is bounded below by $m = 0$.

T **5.** Let M be a given positive integer. Then, $\sum_{n=M}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

Let $N \geq M$

$$\sum_{n=1}^{N} a_n = a_1 + a_2 + \dots + a_{M-1} + \sum_{n=M}^{N} a_n$$

Sum of finite terms $a_1 + a_2 + \dots + a_{M-1}$ is finite

Thus $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$ converges

$\Leftrightarrow \sum_{n=M}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=M}^{N} a_n$ Converges

F **6.** Let $S = 1 - 1 + 1 - 1 + 1 - \dots$. Then $-S = -1 + 1 - 1 + 1 - 1 + \dots = -1 + S$, so we have $S = \frac{1}{2}$.

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + 1 - \cdots$$

令 $S_N = \sum_{n=1}^{N} (-1)^{n-1} \Rightarrow S_1 = 1$

$S_2 = 1 - 1 = 0$

$S_3 = 1 - 1 + 1 = 1$

$S_4 = 1 - 1 + 1 - 1 = 0$

$\Rightarrow S_{2N} = 0$, $S_{2N-1} = 1$ $\forall N = 1, 2, \dots \Rightarrow \lim_{N\to\infty} S_N$ 不存在

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1}$ 發散

$\therefore$ 相關的演算不成立

F **7.** If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=2}^{\infty} \frac{a_n}{n}$ converges.

Counterexample:

Let $a_n = \frac{1}{\ln n} \Rightarrow \lim_{n\to\infty} a_n = 0$

However, $\sum_{n=2}^{\infty} \frac{a_n}{n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by Integral test.

Let $f(x) = \frac{1}{x \ln x} > 0$. decreasing & continuous $\forall x \geq 2$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_{\ln 2}^{\infty} \frac{1}{u} du = \infty \quad \text{diverges}$$

$(u = \ln x.\ du = \frac{1}{x} dx)$

Thus, $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

T **8.** If $\sum_{n=1}^{\infty} a_n$ does not converge, then $\sum_{n=1}^{\infty} |a_n|$ does not converge.

§11.4 Thm 1 Absolute Convergence Implies Convergence.

If $\Sigma|a_n|$ converges, then Σa_n converges.

$\Leftrightarrow$ If Σa_n doesn't converge, then $\Sigma|a_n|$ doesn't converge.

9. If $a_n > 0$ and $\frac{a_{n+1}}{a_n} < 1$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

F Counterexample:

Let $a_n = 1 + \frac{1}{n}$ then $a_{n+1} < a_n$ for all $n \geq 1$

So $\frac{a_{n+1}}{a_n} < 1$ but $\lim_{n\to\infty} a_n = 1 \neq 0$

$\therefore \sum_{n=1}^{\infty} a_n$ is divergent by Divergence test.

10. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$ converges conditionally.

T Since $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$

$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

And $\{\frac{1}{n}\}$ is a decreasing positive sequence that converges to 0, then $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is convergent by Leibniz Test for Alternating Series.

But $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-series test.

(下頁還有試題)

填充題 (40 points)，每題 5 分。(請依題號順序依序寫在答案卷第一頁上)

1. Find the centroid of the top half of the ellipse $\frac{x^2}{4} + y^2 = 1$.

Answer : $\underline{(0, \frac{4}{3\pi})}$.

Let $(\bar{x}, \bar{y})$ be the centroid of the shaded region.
Then $\bar{x} = 0$ by symmetry principle, and
$\bar{y} = \frac{M_x}{M}$, where $M_x = \rho \cdot \int_{-2}^{2} \frac{1}{2} y^2 dx$, and
$M = \rho \cdot \frac{1}{2}$ (area of the ellipse) $= \rho \cdot \pi$.
WLOG, we can assume the density $\rho = 1$. Then

$$M_x = \int_{-2}^{2} \frac{1}{2}\left(1 - \frac{x^2}{4}\right) dx$$

$$= \frac{1}{2}\left(x - \frac{x^3}{12}\right)\Big|_{-2}^{2} = \frac{1}{2}\left(\frac{4}{3} - \left(-\frac{4}{3}\right)\right) = \frac{4}{3}$$

$$\Rightarrow \bar{y} = \frac{M_x}{M} = \frac{\frac{4}{3}}{\pi} = \frac{4}{3\pi}$$

$$\Rightarrow \text{centroid} = \left(0, \frac{4}{3\pi}\right)$$

2. Write down the first three terms of the Maclaurin Series for

$$f(x) = (1 + x)^{\frac{1}{2}} \sin 2x.$$

Answer : $\underline{2x + x^2 - \frac{19}{12}x^3}$

- The binomial series

$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n$ for $|x| < 1$, with coefficients

$\binom{\frac{1}{2}}{0} = 1$, $\binom{\frac{1}{2}}{1} = \frac{1}{2}$, $\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\cdot(-\frac{1}{2})}{1\cdot 2} = -\frac{1}{8}$, ...

$$\Rightarrow (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$ for $x \in \mathbb{R}$

$$\Rightarrow \sin 2x = 2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \cdots$$

- $f(x) = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots\right)\left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \cdots\right)$

$$= 2x + x^2 - \left(\frac{4}{3} + \frac{1}{4}\right)x^3 + \cdots$$

$$= 2x + x^2 - \frac{19}{12}x^3 + \cdots$$

3. A particle travels along a cycloid. Its position at time t is given by $c(t) = (t - \sin t, 1 - \cos t)$ for $t \geq 0$. Find the maximum speed of the particle.

Answer : 2 .

Let $x(t) = t - \sin t$, $y(t) = 1 - \cos t$

$\Rightarrow x'(t) = 1 - \cos t$, $y'(t) = \sin t$

The speed is $\frac{ds}{dt} = \sqrt{(1-\cos t)^2 + (\sin t)^2}$

$= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$

$= \sqrt{2 - 2\cos t}$

Hence the maximum speed of the particle occurs at the value of t where $f(t) = 2 - 2\cos t$ has maximum value 4. ($\because -1 \leq \cos t \leq 1$)

$\Rightarrow$ The maximum speed of the particle is 2.

(For $t = (2n-1)\pi$, $n \in \mathbb{N}$)

4. If $\lim_{n\to\infty} a_n\sqrt{n} = 3$, then find $\lim_{n\to\infty} a_n$. Answer : 0 .

$\lim_{n\to\infty} a_n\sqrt{n} = 3 \overset{def}{\iff} \forall \varepsilon > 0 \ \exists N = N_\varepsilon$ s.t. $\forall n \geq N$, $|a_n\sqrt{n} - 3| < \varepsilon$

In particular.

Choose $\varepsilon = 1$, $\exists N$ s.t

$\forall n \geq N$, $|a_n\sqrt{n} - 3| < 1$

$\Rightarrow 2 < a_n\sqrt{n} < 4$, $\forall n \geq N$

$\Rightarrow \frac{2}{\sqrt{n}} < a_n < \frac{4}{\sqrt{n}}$, $\forall n \geq N$

$0 = \lim_{n\to\infty} \frac{2}{\sqrt{n}} \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \frac{4}{\sqrt{n}} = 0$

Thus $\lim_{n\to\infty} a_n = 0$.

5. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$.

Answer : $\underline{\quad 3 \quad}$. (By § 11.6 Thm 2)

Let $\sum_{n=1}^{\infty} \frac{1}{n^2 3^n} x^n = \sum_{n=1}^{\infty} b_n x^n$, $b_n = \frac{1}{n^2 3^n}$

$$r = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right|$$

$$= \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2 3^{n+1}}}{\frac{1}{n^2 3^n}}$$

$$= \lim_{n\to\infty} \frac{1}{3} \frac{n^2}{(n+1)^2}$$

$$= \frac{1}{3}$$

$\Rightarrow$ radius of convergence $R = \frac{1}{r} = 3$

6. Let $c(t) = (t^2 - 9, t^2 - 8t)$. Find the equation of the tangent line at $t = 4$.

Answer : $\underline{\ y = -16\ }$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{d}{dt}(t^2 - 8t)}{\frac{d}{dt}(t^2 - 9)}$$

$$= \frac{2t - 8}{2t} \bigg|_{t=4} = 0$$

$$C(4) = (4^2 - 9, 4^2 - 8 \cdot 4) = (7, -16)$$

Thus equation of tangent line is

$y = -16$.

7. Find the area of the surfaces generated by revolving the curve $x = \cos t$, $y = 2 + \sin t$, $0 \le t \le 2\pi$ about the $x-$axis.

Answer : $8\pi^2$.

$x'(t) = -\sin t$

$y'(t) = \cos t$

The surface area S

$= 2\pi \int_0^{2\pi} (2+\sin t)\sqrt{(-\sin t)^2+(\cos t)^2}\,dt$

$= 2\pi\,(2t-\cos t)\Big|_0^{2\pi}$

$= 2\pi\,(4\pi-\cancel{1}+\cancel{1}) = 8\pi^2$

8. Find the arc length of $f(x) = \left(\frac{x}{2}\right)^4 + \frac{1}{2x^2}$ over $[2, 4]$.

Answer : $\frac{483}{32}$.

$f'(x) = 4\left(\frac{x}{2}\right)^3 \cdot \frac{1}{2} - x^{-3} = \frac{1}{4}x^3 - x^{-3}$

$\Rightarrow 1 + f'(x)^2 = 1 + \left(\frac{x^3}{4} - x^{-3}\right)^2 = 1 + \frac{x^6}{16} - \frac{1}{2} + x^{-6} = \left(\frac{x^3}{4} + x^{-3}\right)^2$

$\therefore$ Arc length over $[2, 4]$

$= \int_2^4 \sqrt{\left(\frac{x^3}{4}+x^{-3}\right)^2}\,dx$

$= \int_2^4 \frac{x^3}{4} + x^{-3}\,dx$

$= \left(\frac{x^4}{16} - \frac{1}{2x^2}\right)\Big|_{x=2}^{4}$

$= \frac{483}{32}$

(下頁還有試題)

計算問答證明題(*60* points)，每題 10 分，請依題號順序依序寫在答案卷上，可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. (*10* points) Determine if the given series converges or diverges.

a. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n \ln n}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\frac{1}{n}$

Sol.

a. Consider $f(x) = \frac{2}{x \ln x}$ $x \in [2, \infty)$

- Note that both $y = x$ and $y = \ln x$ are increasing functions, so $f(x)$ is decreasing. Hence $f(x)$ is positive, decreasing and continuous on $[2, \infty)$.

- $\displaystyle\int_2^{\infty} \frac{2}{x \ln x}\, dx = \lim_{R\to\infty} \int_2^R \frac{2}{x \ln x}\, dx$

$$(u = \ln x \Rightarrow du = \tfrac{1}{x}dx) \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{2}{u}\, du$$

$$= \lim_{R\to\infty} 2 \ln|u| \Big|_{\ln 2}^{\ln R}$$

$$= \lim_{R\to\infty} \left(2\ln(\ln R) - 2\ln(\ln 2)\right) = \infty$$

By Integral test, $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n \ln n}$ diverges.

b. Since $\displaystyle\lim_{n\to\infty} \cos\frac{1}{n} = 1 \neq 0$, $\displaystyle\lim_{n\to\infty} (-1)^n \cos\frac{1}{n} \neq 0$

(The limit actually does not exist.)

By Divergence test, $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\frac{1}{n}$ diverges.

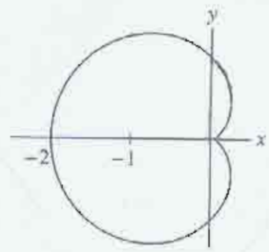
2. (*10* points) **a.** Sketch the curve with the polar equation $r = 1 - \cos\theta$.

b. Find the area of the region enclosed by $r = 1 - \cos\theta$.

Sol.

a.

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{2}$	1
π	2
$\frac{3\pi}{2}$	1
2π	0



b.

$$
\begin{aligned}
\text{Area} &= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta \\
&= \frac{1}{2}\left(\theta - 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} \\
&= \frac{1}{2}(3\pi) \\
&= \frac{3\pi}{2}
\end{aligned}
$$

3. (*10* points) Let $T_n(x)$ be the nth Taylor polynomial of $f(x) = \ln x$ centered at $a = 1$.

a. Find $T_4(x)$.

b. Use the Error bound to find a bound for the error $|\, T_3(1.2) - \ln 1.2 \,|$.

Sol.

a. $T_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$

$f(x) = \ln x$, $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f^{(3)}(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$

$\Rightarrow f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f^{(3)}(1) = 2$, $f^{(4)}(1) = -6$

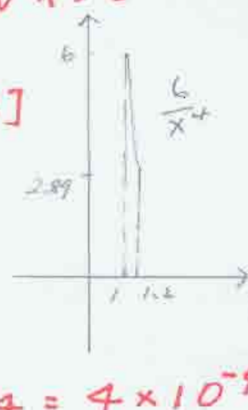
$\Rightarrow T_4(x) = (x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4$

$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

b. By §9.4 Thm1 Error bound

$f^{(4)}(x) = -6x^{-4}$ exists and is continuous $\forall x > 0$

$|f^{(4)}(x)| = \frac{6}{x^4} \le \max_{x \in [1, 1.2]} \frac{6}{x^4} = 6$, $\forall x \in [1, 1.2]$



Thus $|T_3(1.2) - \ln 1.2| \le 6 \frac{|1.2-1|^4}{4!}$

$= \frac{6}{4!}(0.2)^4$

$= \frac{1}{4}(0.2)^4 = 0.0004 = 4 \times 10^{-4}$

4. (*10* points) Find the value of x for which the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{\sqrt{n^2+1}}$ converges.

Let $\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{\sqrt{n^2+1}} = \sum_{n=0}^{\infty} a_n$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{\sqrt{(n+1)^2+1}} \Big/ \frac{(-1)^n x^n}{\sqrt{n^2+1}}\right|$$

$$= |x| \lim_{n\to\infty}\frac{\sqrt{n^2+1}}{\sqrt{(n+1)^2+1}} = |x|$$

Ratio test $\Longrightarrow$

(i) $\rho = |x| < 1 \Rightarrow \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{\sqrt{n^2+1}}$ converges absolutely

$\xRightarrow{\S 11.4\ \text{Thm 1}} \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{\sqrt{n^2+1}}$ converges.

(ii) $\rho = |x| > 1 \Rightarrow \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{\sqrt{n^2+1}}$ diverges.

(iii) $\rho = |x| = 1 \Rightarrow x = 1$ or -1

ⓐ $x = 1 \Rightarrow \sum_{n=0}^{\infty}(-1)^n\frac{1}{\sqrt{n^2+1}}$ converges by Leibniz Test for alternating series.
Since $a_n = \frac{1}{\sqrt{n^2+1}} > 0$, decreasing, $\lim_{n\to\infty} a_n = 0$.

ⓑ $x = -1 \Rightarrow \sum_{n=0}^{\infty}(-1)^n\frac{(-1)^n}{\sqrt{n^2+1}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+1}}$
diverges by limit comparison test.
Since $\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = 1$ & harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Thus $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+1}}$ diverges.

Hence, the power series converges for $-1 < x \le 1$.

5. (*10* points) Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Determine whether the following series are convergent or divergent.

a. $\sum_{n=1}^{\infty} \left(a_n + \frac{1}{n^2}\right)$

b. $\sum_{n=1}^{\infty} \frac{1}{1 + {a_n}^2}$

Sol.

a. (By § 11.2 Thm 3)

$\because \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by P-series test

Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are both convergent

then $\sum_{n=1}^{\infty} (a_n + \frac{1}{n^2}) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \frac{1}{n^2}$ is Convergent.

b. $\because \sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \lim_{n\to\infty} |a_n| = 0$

$\Rightarrow \exists N \in \mathbb{N} \ni |a_n| < 1, \ \forall n \geq N$

$\Rightarrow 0 \leq a_n^2 = |a_n| \cdot |a_n| < |a_n| \cdot 1 = |a_n| \quad \forall n \geq N$

By comparison test $\Rightarrow \sum_{n=1}^{\infty} a_n^2$ converges

$\Rightarrow \lim_{n\to\infty} a_n^2 = 0$

$\therefore \lim_{n\to\infty} \frac{1}{1+a_n^2} = 1 \neq 0$

$\therefore \sum_{n=1}^{\infty} \frac{1}{1+a_n^2}$ is divergent by Divergence Test.

6. (*10* points) **a.** Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

b. Prove $\frac{\pi}{4} = \tan^{-1} 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Sol.

a. Let $\sum_{n=0}^{\infty} \frac{(-1) x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} b_n x^{2n+1}$, $b_n = \frac{(-1)^n}{2n+1}$

$r = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right|$

$= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{2n+3}}{\frac{(-1)^n}{2n+1}} \right| = \lim_{n\to\infty} \frac{2n+1}{2n+3} = 1$

$\Rightarrow$ radius of convergence $R = \frac{1}{r} = 1$.

b. First, we prove that $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $-1 < x < 1$.

$\because \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = 1 - x^2 + x^4 - x^6 + \cdots$

Since the geometric series has radius of convergence 1, this expansion is valid for $|x^2| < 1$, that is, $|x| < 1$.

Now apply §11.6 Thm 3

$\therefore \tan^{-1} x = \int \frac{1}{1+x^2} dx = \int 1 - x^2 + x^4 - x^6 + \cdots dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

As $x = 0 \Rightarrow C = \tan^{-1} 0 = 0$

$\therefore \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $-1 < x < 1$

Then, for the endpoint $x = 1$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

$\because \left\{\frac{1}{2n+1}\right\}$ is a decreasing positive sequence that converges to 0

$\therefore \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ is convergent by Leibniz test for Alternating series

So $\lim_{x\to 1^-} \tan^{-1} x = \lim_{x\to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ (The limit exists for the above reason!)

$\Rightarrow \tan^{-1} 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

and $\tan^{-1} 1 = \frac{\pi}{4}$

Thus $\frac{\pi}{4} = \tan^{-1} 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

(試題結束)