微積分聯合教學　段考五　　May 11, 2010

考試時間 120 分鐘，題目卷為兩張紙，共三頁，滿分 120 分。所有題目的答案都請依題號順序依序寫在答案卷上，而是非與填充題必須寫在第一頁。答案卷務必寫學號、姓名，題目卷不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘前不得離場。考試期間禁止使用字典、計算機及任何通訊器材，監試人員不得回答任何關於試題的疑問。

是非題 (*30* points)，請答 **T** (True) 或 **F** (False)。每題 3 分。(請依題號順序依序寫在答案卷第一頁上)

1. $\mathbf{r}(t) = t^3\mathbf{i} - 7t^3\mathbf{j} + t^3\mathbf{k}$ is a parametrization of a line.

T　It's convenient to represent the particle's path by the vector-valued function

$$r(t) = \langle x(t), y(t), z(t)\rangle = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.$$

In this case, $x(t) = t^3$, $y(t) = -7t^3$ and $z(t) = t^3$.

2. Two different level curves of $f(x, y)$ never intersect.

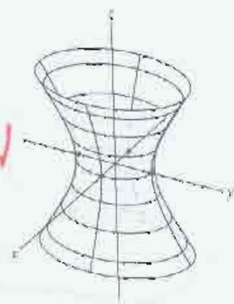
T　Suppose the curves intersect at (x_0, y_0), then $f(x_0, y_0) = C_1$ and $f(x_0, y_0) = C_2$.

$\Rightarrow C_1 = C_2 \quad \rightarrow\leftarrow$

Therefore, two different level curves of $f(x,y)$ never intersect.

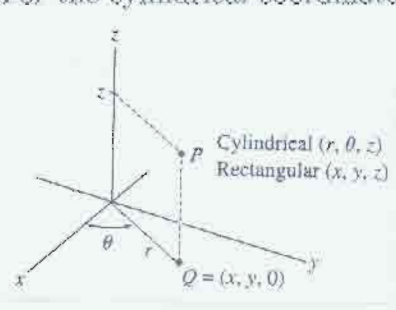
3. All traces of a hyperboloid are hyperbolas.

F　From the figure, we observe that the horizontal traces are ellipse.

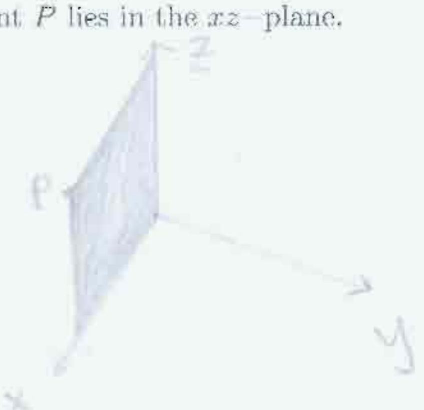


4. For the cylindrical coordinates, if $\theta = 0$ then the point P lies in the xz-plane.

T

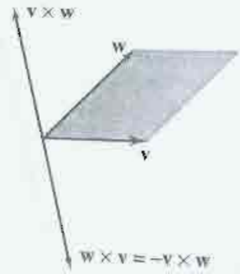


$\theta = 0$ $\Longrightarrow$



5. Let $\mathbf{v}$ and $\mathbf{w}$ be vectors. $\mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{v}$.

F $V \times W$ and $W \times V$ point in opposite directions by the right-hand rule and thus $V \times W = -W \times V$.



6. The curvature of a circle of radius R is $1/R$.

T Refer to the example 2 of section 14.4 in your textbook. (page 744)

7. Let $f(x, y)$ be a function of two variables. If the partial derivative $f_x(a, b)$ and $f_y(a, b)$ exist, $f(x, y)$ is differentiable at (a, b).

F Let $f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

$\Rightarrow f_x(0,0) = 0, \; f_y(0,0) = 0$

but $f(x,y)$ is not continuous at $(0,0)$

$\Rightarrow f(x,y)$ is not differentiable at $(0,0)$

8. $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{w}$ is orthogonal to $\mathbf{u} \times \mathbf{v}$.

F Counterexample:

let $U = \langle 1, 4, 5 \rangle$, $V = \langle -2, -1, 2 \rangle$

$W = \langle 26, -24, 14 \rangle$

$\Rightarrow W \perp U$ and $W \perp V$

And $U \times V = \langle 13, -12, 7 \rangle$

But $W \parallel U \times V$.

9. $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal to each other.

F Refer to the example 6 of section 14.2 in your textbook. (page 732)

Counterexample:

$\vec{r}(t) = (t, t)$, $\vec{r}'(t) = (1, 1)$

$\vec{r}(t) \cdot \vec{r}'(t) = 2t \neq 0$ if $t \neq 0$

10. A differentiable function increases at the rate $\|\nabla f_p\|$ in the direction of ∇f_p.

T $D_u f(p) = \|\nabla f_p\| \cos\theta$, where θ is the angle between u and ∇f_p.

If u is the direction of ∇f_p then $\theta = 0$ and $D_u f(p) = \|\nabla f(p)\|$.

填充題 (*40* points)，每題 5 分。(請依題號順序依序寫在答案卷第一頁上)

1. Compute the derivative $g_{yyx}(1, 0)$ with $g(x, y) = xye^{-y}$.

Answer : -2.

$\because g_y(x, y) = xe^{-y}(1-y)$

$g_{yy}(x, y) = xe^{-y}(y-2)$

$g_{yyx}(x, y) = e^{-y}(y-2)$

$\therefore g_{yyx}(1, 0) = e^{-0}(0-2) = -2$

2. Assume that $f(2, 3) = 8$, $f_x(2, 3) = 5$, and $f_y(2, 3) = 7$. Estimate $f(2, 3.1)$.

Answer : 8.7.

The linear approximation at $(2, 3)$ is

$f(2+h, 3+k) \approx f(2,3) + f_x(2,3)h + f_y(2,3)k$

$\approx 8 + 5h + 7k$

$\Rightarrow f(2, 3.1) \approx 8 + 5 \cdot 0 + 7 \cdot 0.1$

≈ 8.7

3. What is the largest value that the directional derivative of $f(x,y,z) = xyz$ can have at the point $(1,1,1)$? Answer : $\underline{\sqrt{3}}$.

By the properties of the dot product,

$D_u f(1,1,1) = \nabla f_P \cdot u = \|\nabla f_P\| \cos\theta$, where u is unit vector and θ is the angle between $\nabla f(1,1,1)$ and u.

Therefore, $D_u f(1,1,1)$ has the largest possible value when $\theta = 0$, that is, when u points in the direction of ∇f_P.

$\because \nabla f = \langle yz, xz, xy \rangle \Rightarrow \nabla f(1,1,1) = \langle 1, 1, 1 \rangle$

$\therefore D_u f(1,1,1) = \sqrt{1^2+1^2+1^2} \cdot \cos 0 = \sqrt{3}$

4. Find the area of the triangle with vertices $P(1,1,5)$, $Q(3,4,3)$ and $R(1,5,7)$.

Answer : $\underline{\sqrt{69}}$.

$\overrightarrow{PQ} = (2, 3, -2)$

$\overrightarrow{PR} = (0, 4, 2)$

$\triangle PQR = \frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\|$

$= \frac{1}{2} \|(14, -4, 8)\|$

$= \sqrt{69}$

5. Find the decomposition $\mathbf{a} = \mathbf{a}_{\|} + \mathbf{a}_{\perp}$ with respect to $\mathbf{b}$ where $\mathbf{a} = \langle 3, 1, 1 \rangle$ and $\mathbf{b} = \langle 5, 2, 1 \rangle$. Answer : __________.

$\vec{a} \cdot \vec{b} = \langle 3, 1, 1 \rangle \cdot \langle 5, 2, 1 \rangle = 18$

$\vec{b} \cdot \vec{b} = \langle 5, 2, 1 \rangle \cdot \langle 5, 2, 1 \rangle = 30$

$\vec{a}_{\|} = \left(\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \right) \vec{b} = \frac{\cancel{18}^{3}}{\cancel{30}_{5}} \langle 5, 2, 1 \rangle = \langle 3, \frac{6}{5}, \frac{3}{5} \rangle$

$\vec{a}_{\perp} = \vec{a} - \vec{a}_{\|}$

$= \langle 3, 1, 1 \rangle - \langle 3, \frac{6}{5}, \frac{3}{5} \rangle = \langle 0, -\frac{1}{5}, \frac{2}{5} \rangle$

$\Rightarrow \vec{a} = \vec{a}_{\|} + \vec{a}_{\perp} = \langle 3, \frac{6}{5}, \frac{3}{5} \rangle + \langle 0, -\frac{1}{5}, \frac{2}{5} \rangle$

6. Find the intersection of the line $\mathbf{r}(t) = \langle 2, -1, -1\rangle + t\langle 1, 2, -4\rangle$ and the plane $2x + y = 3$. Answer : $(2, -1, -1)$.

$\because$ The parametric equations of the line are

$x = 2 + t$, $y = -1 + 2t$, $z = -1 - 4t$

$\therefore 2x + y = 3 \Rightarrow 2(2+t) + (-1+2t) = 3$

$\Rightarrow 4t = 0$

$\Rightarrow t = 0$

Thus $x = 2$, $y = -1$, $z = -1$.

7. Let $\mathbf{r}(t) = \langle 3t + 1, 4t - 5, 2t\rangle$. Find the arc length parametrization of $\mathbf{r}(t)$.

Answer : $< \frac{3s}{\sqrt{29}} + 1, \frac{4s}{\sqrt{29}} - 5, \frac{2s}{\sqrt{29}} >$

$\because r'(t) = <3, 4, 2> \rightarrow \|r'(t)\| = \sqrt{29}$

$\therefore$ the arc length function is

$s(t) = \int_0^t \|r'(t)\| dt = \int_0^t \sqrt{29}\, dt = \sqrt{29}\, t$

The inverse of $s(t) = \sqrt{29}\,t$ is $t = \varphi(s) = \frac{s}{\sqrt{29}}$

$\therefore$ We obtain the following arc length parametrization

$r_1(s) = r(\frac{s}{\sqrt{29}}) = < \frac{3s}{\sqrt{29}} + 1, \frac{4s}{\sqrt{29}} - 5, \frac{2s}{\sqrt{29}} >$

As a check, we verify that $r_1(s)$ has unit speed:

$r_1'(s) = \frac{1}{\sqrt{29}} <3, 4, 2>$ and $\|r_1'(s)\| = \frac{1}{\sqrt{29}} \|<3, 4, 2>\| = 1$

8. Find the directional derivative of $f(x, y, z) = xy + z^3$ at $p = (3, -2, -1)$ in the direction pointing to the origin. Answer : $\frac{15}{\sqrt{14}}$.

The direction vector $V = \overrightarrow{PO} = <-3, 2, 1>$

$\Rightarrow u = \frac{V}{\|V\|} = \frac{1}{\sqrt{14}} <-3, 2, 1>$

$\because \nabla f = <y, x, 3z^2> \Rightarrow \nabla f_{(3,-2,-1)} = <-2, 3, 3>$

$\therefore$ the directional derivative is

$D_u f_{(3,-2,-1)} = \nabla f_{(3,-2,-1)} \cdot u = <-2, 3, 3> \cdot \frac{1}{\sqrt{14}} <-3, 2, 1>$

(下頁還有試題) $= \frac{1}{\sqrt{14}} (6 + 6 + 3)$

$= \frac{15}{\sqrt{14}}$

計算問答證明題($\mathit{50}$ points)，每題 10 分，請依題號順序依序寫在答案卷上，可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. ($\mathit{10}$ points) Find an equation of the plane determined by the points $(1, 0, -1)$, $(2, 2, 1)$, and $(4, 1, 2)$.

Let $P = (1, 0, -1)$
$Q = (2, 2, 1)$
$R = (4, 1, 2)$

$\therefore \overrightarrow{PQ} = \langle 1, 2, 2 \rangle$

$\overrightarrow{PR} = \langle 3, 1, 3 \rangle$

$n = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 3 & 1 & 3 \end{vmatrix} = \langle 4, 3, -5 \rangle$

Thus the equation of the plane is

$4x + 3y - 5z = d$,

where $d = n \cdot \overrightarrow{op} = \langle 4, 3, -5 \rangle \cdot \langle 1, 0, -1 \rangle$
$= 9$

$\therefore$ The equation is $4x + 3y - 5z = 9$.
(or $4(x-1) + 3(y-0) - 5(z+1) = 0$)

2. (*10* points) Let $f(x,y) = \dfrac{xy^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$. Is it possible to define $f(0,0)$ in a way that makes f continuous at the origin? Explain your answer and show your reasons. (*Hint:* Let $x = r\cos\theta$ and $y = r\sin\theta$.)

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$$= \lim_{r\to 0} \frac{r\cos\theta(r\sin\theta)^2}{\cancel{(r\cos\theta)^2+(r\sin\theta)^2}\ r^2}$$

$$= \lim_{r\to 0} r\cos\theta\sin^2\theta$$

$$= 0$$

$\because \lim_{(x,y)\to(0,0)} f(x,y) = 0$

$\therefore$ We define $f(0,0) = 0$

Then $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

is continuous at the origin.

3. (*10* points) Find an equation of tangent plane to the surface $x^2 + 3y^2 + 4z^2 = 20$ at $P = (2, 2, 1)$.

Let $F(x, y, z) = x^2 + 3y^2 + 4z^2$.

Then $\nabla F = \langle 2x, 6y, 8z \rangle$

$\nabla F_P = \nabla F_{(2,2,1)} = \langle 4, 12, 8 \rangle$

The vector ∇F_P is normal to $F(x, y, z) = 20$.

So the tangent plane has equation

$4(x-2) + 12(y-2) + 8(z-1) = 0$

i.e $x + 3y + 2z = 10$

4. (*10* points) Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a path with curvature $k(t)$ and define the scaled path $\mathbf{r}_1(t) = \langle \lambda x(t), \lambda y(t), \lambda z(t)\rangle$, where $\lambda > 0$ is a constant. $k_1(t)$ is the curvature of $\mathbf{r}_1(t)$. Prove that $k_1(t) = \dfrac{1}{\lambda}k(t)$.

By the Theorem 1 in Section 14.4 (page 754)

If $\vec{r}(t)$ is a regular parametrization, then the curvature at $\vec{r}(t)$ is $k(t) = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$

We use the above equation:

$$k_1(t) = \frac{\|r_1'(t) \times r_1''(t)\|}{\|r_1'(t)\|^3} \quad \text{and} \quad k(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$\because$ $r_1'(t) = \lambda r'(t)$

$r_1''(t) = \lambda r''(t)$

$\therefore$ $\|r_1'(t) \times r_1''(t)\| = \|\lambda r'(t) \times \lambda r''(t)\| = \lambda^2 \|r'(t) \times r''(t)\|$

$\|r_1'(t)\| = |\lambda| \|r'(t)\|$

Thus we obtain

$$k_1(t) = \frac{\|r_1'(t) \times r_1''(t)\|}{\|r_1'(t)\|^3} = \frac{\lambda^2 \|r'(t) \times r''(t)\|}{|\lambda|^3 \|r'(t)\|^3}$$

$$= \frac{1}{|\lambda|} \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$$= \frac{1}{|\lambda|} k(t) \underset{(\lambda > 0)}{=} \frac{1}{\lambda} k(t).$$

5. (*10* points) Find the points on the graph of $z = 3x^2 - 4y^2$ at which the vector $\mathbf{n} = \langle 3, 2, 2 \rangle$ is normal to the tangent plane and the equation of that tangent plane.

Let $P = (a, b, f(a,b))$

$\Rightarrow$ the tangent plane at P on $z = f(x,y)$ is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$\Rightarrow f_x(a,b)(x-a) + f_y(a,b)(y-b) - z + f(a,b) = 0$

Let $V = \langle f_x(a,b), f_y(a,b), -1 \rangle$ be the normal vector to the tangent plane at P

Since we want $n = \langle 3, 2, 2 \rangle$ to be normal to the plane

$\therefore V \parallel n$

$\because f_x(x,y) = 6x \Rightarrow f_x(a,b) = 6a$

$f_y(x,y) = -8y \Rightarrow f_y(a,b) = -8b$

$\therefore \frac{6a}{3} = \frac{-8b}{2} = \frac{-1}{2}$

$\Rightarrow a = -\frac{1}{4},\ b = \frac{1}{8}$

$\therefore z = 3(-\frac{1}{4})^2 - 4(\frac{1}{8})^2 = \frac{1}{8}$

Thus $P = (-\frac{1}{4}, \frac{1}{8}, \frac{1}{8})$ and the equation of the tangent plane is $3(x+\frac{1}{4}) + 2(y-\frac{1}{8}) + 2(z-\frac{1}{8}) = 0$

(i.e $3x + 2y + 2z = -\frac{1}{4}$)

(試題結束)