Calculus Homework Assignment 1

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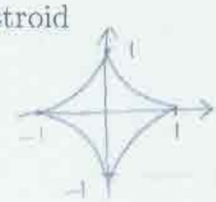
1. Calculate the length of the astroid

$$x^{2/3} + y^{2/3} = 1.$$



[§9.1 #15]

By implicit differentiation

$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0 \Rightarrow y' = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

$1+(y')^2 = 1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{1}{x^{\frac{2}{3}}}$

$S = \int_0^1 \sqrt{1+(y')^2}\,dx = \int_0^1 \frac{1}{x^{\frac{1}{3}}}\,dx = \frac{3}{2}x^{\frac{2}{3}}\Big|_0^1$

$= \frac{3}{2}$

The total arc length is $4 \cdot \frac{3}{2} = 6$

2. Compute the surface area of revolution

$$y = e^x = f(x)$$

about the $x-$axis over the interval $[0, 1]$.

[§9.1 #37] $SA = 2\pi\int_0^1 f(x)\sqrt{1+[f'(x)]^2}\,dx$

Let $y = e^x$. Then $y' = e^x$ and

SA = Surface area $= 2\pi\int_0^1 e^x\sqrt{1+e^{2x}}\,dx$

Using the substitution $(1+\tan^2\theta = \sec^2\theta)$

$e^x = \tan\theta$. $e^x dx = \sec^2\theta\, d\theta$

$\int e^x\sqrt{1+e^{2x}}\,dx = \int \sec^3\theta\, d\theta$ (P434 L2↓)

$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$

$= \frac{1}{2}e^x\sqrt{1+e^{2x}} + \frac{1}{2}\ln|\sqrt{1+e^{2x}}+e^x| + C$

Finally, surface area $= (\pi e^x\sqrt{1+e^{2x}} + \pi\ln|\sqrt{1+e^{2x}}+e^x|)\Big|_0^1$

$= \pi e\sqrt{1+e^2} + \pi\ln(\sqrt{1+e^2}+e) - \pi\sqrt{2} - \pi\ln(\sqrt{2}+1)$

$= \pi e\sqrt{1+e^2} - \pi\sqrt{2} + \pi\ln\left(\frac{\sqrt{1+e^2}+e}{\sqrt{2}+1}\right)$

3. Find the centroid of the top half of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ for the arbitary $a, b > 0$.

[§9.3 #29] Centroid i.e. COM of a lamina of constant mass density ρ. WLOG $\rho = 1$

The equation of top half of the ellipse

$y = \sqrt{b^2 - \frac{b^2x^2}{a^2}}$

$M_x = \frac{1}{2}\int_{-a}^{a}\left(\sqrt{b^2-\frac{b^2x^2}{a^2}}\right)^2 dx$

$= \frac{1}{2}\left(b^2x - \frac{b^2x^3}{3a^2}\right)\Big|_{-a}^{a}$

$= \frac{1}{2}\left(b^2a - \frac{b^2a}{3} - b^2(-a) + \frac{b^2(-a)}{3}\right)$

$= \frac{1}{2}\left(2ab^2 - \frac{2ab^2}{3}\right) = \frac{2}{3}ab^2$

By symmetric principle, $x_{CM} = 0$.

area of a half ellipse $M = \frac{\pi ab}{2} \Rightarrow y_{CM} = \frac{M_x}{M} = \frac{4b}{3\pi}$

4. Let $f(x) = \sqrt{1+x}$ and let $T_n(x)$ be the Taylor polynomial centered at $a = 3$. Find $T_3(x)$ and calculate $T_3(3.02)$.

[like §9.4 #32(a)]

$f(x) = \sqrt{1+x}$ $\quad f(3) = 2$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\quad f'(3) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ $\quad f''(3) = -\frac{1}{32}$

$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$ $\quad f'''(3) = \frac{3}{256}$

$\Rightarrow T_3(x) = 2 + \frac{1}{4}(x-3) - \frac{1}{32\cdot 2!}(x-3)^2 + \frac{3}{256\cdot 3!}(x-3)^3$

$= 2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2 + \frac{1}{512}(x-3)^3$

$\Rightarrow T_3(3.02)$

$= 2 + \frac{1}{4}(0.02) - \frac{1}{64}(0.02)^2 + \frac{1}{512}(0.02)^3$

$= 2.004993765625$

(Over Please)

5. Show that the Maclaurin polynomials for $f(x) = \sin x$ are

$$T_{2n-1}(x) = T_{2n} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

Use the Error Bound with $n = 4$ to show taht $|\sin x - \left(x - \frac{x^3}{6}\right)| \leq \frac{|x|^5}{120}$ (for all x) [§9.4 #35]

$f(x) = \sin x \quad f(0) = 0$

$f'(x) = \cos x \quad f'(0) = 1$

$f''(x) = -\sin x \quad f''(0) = 0$

$f'''(x) = -\cos x \quad f'''(0) = -1$

$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$

$\vdots$

Thus $f^{(2k)}(0) = 0, \ k = 0, 1, \cdots$

$f^{(2k-1)}(0) = (-1)^{k+1}, \ k = 1, 2, \cdots$

$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j$

$T_{2n-1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$

and $T_{2n}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} + 0 = T_{2n-1}(x)$

With $n = 4$. $|f^{(4+1)}(x)| = |\cos x| \leq 1 = K$, for all x.

By P496 Thm 2,

$|T_4(x) - f(x)| = |(x - \frac{x^3}{3!}) - \sin x| \leq 1 \cdot \frac{|x|^5}{5!}$ for all x.

6. Find

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

[§11.1 #35]

Let $a_n = (1 + \frac{1}{n})^n$

taking $\ln \Rightarrow \ln a_n = n \ln(1 + \frac{1}{n}) = \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}}$

$\lim_{n\to\infty}(\ln a_n) = \lim_{n\to\infty}\left(\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}}\right) = \lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}$

$\overset{\text{L'H}}{\underset{0/0}{=}} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot(-\frac{1}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$

Because $f(x) = e^x$ is a continuous function

it follows that

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{\lim_{n\to\infty}(\ln a_n)} = e^1 = e$

7. Determine the limit of the sequence

$$B_n = \frac{n!}{2^n}$$

or show that the sequence diverges by using the appropriate Limit Laws or theorems. [§11.1 #60]

$\frac{n!}{2^n} = \frac{1\cdot 2 \cdot \cdots \cdot n}{2\cdot 2 \cdots 2} = \frac{1}{2}\cdot\frac{2}{2}\cdot\left(\frac{3}{2}\,\frac{4}{2}\cdots\frac{n-1}{2}\right)\frac{n}{2}$

$= \left(\frac{3}{2}\cdot\frac{4}{2}\cdots\frac{n-1}{2}\right)\cdot\frac{n}{4}$

We have $\frac{n!}{2^n} > \frac{n}{4}$

Since $\lim_{n\to\infty}\frac{n}{4} = \infty$,

the Squeeze Thm for sequences implies that $\lim_{n\to\infty}\frac{n!}{2^n} = \infty$

8. Let $\{a_n\}$ be the sequence

$$\sqrt{2}, \quad \sqrt{2\sqrt{2}}, \quad \sqrt{2\sqrt{2\sqrt{2}}}, ...$$

Show that $\{a_n\}$ is increasing and $0 \leq a_n \leq 2$. Then prove that $\{a_n\}$ convergences and find the limit. Seq. is defined recursively, [§11.1 #76]

$a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2a_n}$

$\Rightarrow a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}} > \sqrt{2} = a_1$

$\Rightarrow a_3 = \sqrt{2a_2} > \sqrt{2a_1} = a_2$

if assume $a_k > a_{k-1}$, then $a_{k+1} = \sqrt{2a_k} > \sqrt{2a_{k-1}} = a_k$

by induction, $a_{n+1} > a_n \ \forall n$

that is, $\{a_n\}$ is increasing.

$\because a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$, it follows that $a_n \geq 0 \ \forall n$

Now $a_1 = \sqrt{2} < 2$.

if $a_k \leq 2$, than $a_{k+1} = \sqrt{2a_k} \leq \sqrt{2\cdot 2} = 2$

by induction, $a_n \leq 2 \ \forall n$.

Since $\{a_n\}$ is increasing and bounded,

by the Thm on Bounded Monotonic sequences (P541 Thm 6)

that this seq $\{a_n\}$ is converging

let $L = \lim_{n\to\infty} a_n$.

$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\sqrt{2a_n} = \sqrt{2}\sqrt{\lim_{n\to\infty} a_n} = \sqrt{2L}$

Thus, $L^2 = 2L$, so $L = 0$ or $L = 2$

Because $\{a_n\}$ is increasing,

we have $a_n \geq a_1 = \sqrt{2}$, $\forall n$

hence the limit satisfy $L \geq \sqrt{2}$.

that is $\lim_{n\to\infty} a_n = 2$ //