Calculus Homework Assignment 2

Class: I c

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1. Explain why each of the following statements is incorrect.

a. If the general term a_n tends to zero, $\sum_{n=1}^{\infty} a_n = 0$

b. If a_n tends to zero, then $\sum_{n=1}^{\infty} a_n$ converges.

c. If a_n tends to L, then $\sum_{n=1}^{\infty} a_n = L$.

[§11.2 #34]

a. If the general term $a_n \to 0$ the series may or may not conv. Even if the series conv. it may not conv. to zero (ex: $a_n = \frac{1}{n(n+1)} \Rightarrow \sum_{n=1}^{\infty} a_n = 1$)

b. ex: $a_n = \frac{1}{\sqrt{n}}$.

2. Find the total length of the infinite zigzag path in Figure 4 as shown in page 555 (each zag occurs at an angle of $\frac{\pi}{4}$). [§11.2 #45]

c. If $L \neq 0$: diverge by Divergence Test

If $L = 0$: ex: $a_n = \frac{1}{n(n+1)}$

3. Use the Integral Test to determine whether the infinite series $\sum_{n=1}^{\infty} ne^{-n^2}$ is convergent. [§11.3 #9]

Let $f(x) = x \cdot e^{-x^2}$, is conti. and positive on interval $x \geq 1$

$f'(x) = e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x) = e^{-x^2}(1 - 2x^2)$

$\Rightarrow f'(x) < 0$ for $x \geq 1 \Rightarrow f \searrow$ for $x \geq 1$

Let $u = x^2 \Rightarrow du = 2x\,dx$

$\Rightarrow \int_1^{\infty} x e^{-x^2} dx = \lim_{R \to \infty} \int_1^R x e^{-x^2} dx$

$= \frac{1}{2} \int_1^{R^2} e^{-u} du = -\frac{1}{2} \lim_{R \to \infty} (e^{-R^2} - e^{-1}) = \frac{1}{2e}$

$\because$ integral conv.

$\Rightarrow \sum_{n=1}^{\infty} ne^{-n^2}$ conv.

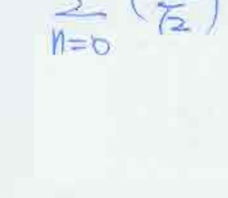
(by Integral Test).

4. Use the Limit Comparison Test to prove convergence or divergence of the infinite series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}.$$

[§11.3 #40]

2.



$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1 - \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2} - 1} = 2 + \sqrt{2}$

4. Let $a_n = \frac{\ln n}{n^2}$

choose $b_n = \frac{1}{n^{\frac{3}{2}}}$

$\Rightarrow L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\ln n}{n^{\frac{1}{2}}}$

$\overset{L}{=} \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{1}{2} n^{-\frac{1}{2}}} = \lim_{n \to \infty} \frac{2}{\sqrt{n}} = 0$

$\because \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ conv. (by P-series)

by the Limit Comparison Test

$\Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ conv.

5. Determine convergence or divergence of the series
$$\sum_{n=2}^{\infty} \frac{1}{n^{1/2} lnn},$$
using any method covered so far. [§11.3 #57]

By L'Hopital's Rule.

$\lim_{x\to\infty} \frac{x^{\frac{1}{4}}}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{4}x^{-\frac{3}{4}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{4}x^{\frac{1}{4}} = \infty$

$\Rightarrow \exists N \in \mathbb{N} \ni \forall n \geq N,\ \frac{n^{\frac{1}{4}}}{\ln n} \geq 1$

$\Rightarrow \frac{1}{n^{\frac{1}{2}} \ln n} \geq \frac{1}{n^{\frac{3}{4}}}$

By p-series $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{4}}}$ div. $\Rightarrow \sum_{n=N}^{\infty} \frac{1}{n^{\frac{3}{4}}}$ div.

By comparison Test $\Rightarrow \sum_{n=N}^{\infty} \frac{1}{n^{\frac{1}{2}} \ln n}$ div.

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}} \cdot \ln n}$ also div. $\divideontimes$

6. Prove that if $\sum a_n$ converges absolutely, then $\sum {a_n}^2$ also converges. Then show by giving a counterexample that $\sum {a_n}^2$ need not converge if $\sum a_n$ is only conditionally convergent. [§11.4 #30]

$\because \sum |a_n|$ conv. $\Rightarrow \lim_{n\to\infty} |a_n| = 0$.

$\Rightarrow \exists N \in \mathbb{N} \ni |a_n| < 1,\ \forall n \geq N.$

$\Rightarrow \underline{0 \leq a_n^2} = |a_n| \cdot |a_n| < |a_n| \cdot 1 = \underline{|a_n|}$

By Comparison Test $\Rightarrow \sum a_n^2$ conv $\divideontimes$

Let $a_n = \frac{(-1)^n}{\sqrt{n}} \Rightarrow \sum_{n=1}^{\infty} a_n$ conv.

(by Leibniz Test)

but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ div. (by p-series)

that is $\sum_{n=1}^{\infty} a_n$ is conditionally conv.

but $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ div. (by p-series).

7. Show that $\sum_{n=1}^{\infty} \frac{r^n}{n}$ converges if $|r| < 1$. [§11.5 #23]

Let $a_n = \frac{r^n}{n}$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{n+1} \cdot \frac{n}{|r|^n} = |r| \cdot \frac{n}{n+1}$

and $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot |r| = |r| < 1$

by Ratio Test

$\Rightarrow \sum_{n=1}^{\infty} \frac{r^n}{n}$ conv. if $|r| < 1$.

8. Determine convergence or divergence using any method covered in the text so far.

a. $\sum_{n=4}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$

b. $\sum_{n=1}^{\infty} (-1)^n \cos \frac{1}{n}$ [§11.5 #39, 48]

a. Let $a_n = (1 + \frac{1}{n})^{-n^2}$

$\Rightarrow \sqrt[n]{a_n} = (1 + \frac{1}{n})^{-n}$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = e^{-1} < 1$

by Root test $\Rightarrow \sum a_n$ conv.

b. $\because \lim_{n\to\infty} \cos \frac{1}{n} = 1 \neq 0$

$\Rightarrow \lim_{n\to\infty} (-1)^n \cos \frac{1}{n} \neq 0$

by Divergence Test

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \cos \frac{1}{n}$ div.