# Calculus Homework Assignment 3

Class: ______________________

Student Number: ______________________

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**1.** Find the values of $x$ for which the following power series converge.

**a.** $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$ **b.** $\displaystyle\sum_{n=12}^{\infty} e^n (x-2)^n$

[§11.6 #11, 23]

a. With $a_n = \frac{1}{\ln n}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\ln n}{\ln(n+1)}$ and $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

The radius of convergence is $R = r^{-1} = 1$.

the series converges absolutely on $|x|<1$.

$x=1$, series diverges by the Comparison Test.

$x=-1$, series converges by the Leibniz Test.

Thus, the series $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$ converges for

$-1 \le x < 1$, and diverges elsewhere. ※

b. With $a_n = e^n$, $\left|\frac{a_{n+1}}{a_n}\right| = e$ and $r = e$,

the series converges absolutely on $|x-2| < e^{-1}$

$x = 2 + e^{-1}$, series diverges by Divergence Test.

$x = 2 - e^{-1}$, series diverges by Divergence Test.

Thus, the series $\sum_{n=12}^{\infty} e^n (x-2)^n$ converges for

$2 - e^{-1} < x < 2 + e^{-1}$ and diverges elsewhere. ※

**2.** Use Eq.(1) in Page 579 to expand the function

$$f(x) = \frac{1}{3-x}$$

in a power series with center $c = 0$ and determine the set of $x$ for which the expansion is valid. [§11.6 #29]

$$\frac{1}{3-x} = \frac{1}{3} \times \frac{1}{1-\frac{x}{3}}$$
$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$$

This series is valid for $\left|\frac{x}{3}\right| < 1$ ※

**3.** Find the Taylor series centered at $c$.

$$f(x) = \frac{1}{1-x^2}, \quad c = 3$$

[§11.7 #39]

3. $$\frac{1}{1-x^2} = \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x} = \frac{\frac{1}{2}}{-2-(x-3)} + \frac{\frac{1}{2}}{4+(x-3)}$$
$$= -\frac{1}{4} \times \frac{1}{1+\frac{x-3}{2}} + \frac{1}{8} \times \frac{1}{1+\frac{x-3}{4}}$$
$$= -\frac{1}{4} \sum_{n=0}^{\infty} \left(-\frac{x-3}{2}\right)^n + \frac{1}{8} \sum_{n=0}^{\infty} \left(-\frac{x-3}{4}\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+2}} (x-3)^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+3}} (x-3)^n$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2^{n+1}-1)}{2^{2n+3}} (x-3)^n$$

This series is valid for $|x-3| < 2$ ※

**4.** Use the Maclaurin expansion for $e^{-t^2}$ to express $\int_0^x e^{-t^2}\,dt$ as an alternating power series in $t$. How many terms of the infinite series are needed to approximate the integral for $x = 1$ to within an error of at most 0.001?

[§11.7 #47(a)]

$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$, $\int_0^x e^{-t^2} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}$

For $x=1$, $\int_0^1 e^{-t^2} dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(2n+1)}$ is an

alternating series with $a_n = \frac{1}{n!(2n+1)}$

The error incurred by using $S_N$ to approximate

the value of the definite integral is bounded

by $\left|\int_0^1 e^{-t^2} dt - S_N\right| \le a_{N+1} = \frac{1}{(N+1)!(2N+3)} < 0.001$

when $N=3$, $(N+1)!(2N+3) = 216 < 1000$

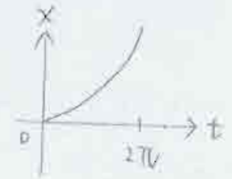
$N=4$, $(N+1)!(2N+3) = 1320 > 1000$

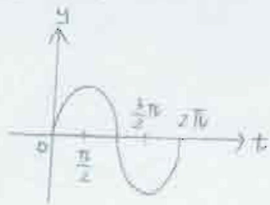
The smallest acceptable value for $N$ is $N=4$ ※

(Over Please) $\left( S_4 = 1 - \frac{1}{3} + \frac{1}{2!\cdot 5} - \frac{1}{3!\cdot 7} + \frac{1}{4!\cdot 9} \right)$

**5.** Sketch $c(t) = (t^2, \sin t)$ for $0 \le t \le 2\pi$.
[like §12.1 #43]

$x(t)=t^2,\ 0\le t\le 2\pi$ $\qquad$ $y(t)=\sin t,\ 0\le t\le 2\pi$




$c(0)=(0,0)$

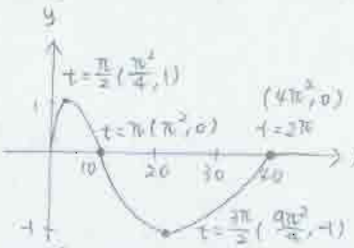
The x-coordinate increases as t increases so the curve is directed to the right. The y-coordinate is positive and increasing for $0<t<\frac{\pi}{2}$, positive and decreasing for $\frac{\pi}{2}<t<\pi$, negative and decreasing for $\pi<t<\frac{3\pi}{2}$, and negative and increasing for $\frac{3\pi}{2}<t<2\pi$.

$c(\frac{\pi}{2})=(\frac{\pi^2}{4},0)$

$c(\pi)=(\pi^2,0)$

$c(\frac{3\pi}{2})=(\frac{9}{4}\pi^2,-1)$

$c(2\pi)=(4\pi^2,0)$




**6.** Let $c(t) = (t^2 - 9, t^2 - 8t)$ (see Figure 17 in Page 612).

**a.** Find the equation of the tangent line at $t = 4$.

**b.** Find the points where the tangent has slope $\frac{1}{2}$. [§12.1 #54, 55]

a. $m=\frac{dy}{dx}\Big|_{t=4}=\frac{(t^2-8t)'}{(t^2-9)'}\Big|_{t=4}=\frac{2t-8}{2t}\Big|_{t=4}$

$=\frac{8-8}{8}=0$

$\Rightarrow$ the tangent line is horizontal

$c(4)=(7,-16)$

$\therefore$ equation is $y=-16$ ※

b. $m=\frac{dy}{dx}=\frac{2t-8}{2t}=\frac{t-4}{t}=\frac{1}{2}$

$\Rightarrow 2t-8=t,\ t=8$

$c(8)=(64-9,64-64)=(55,0)$ ※

**7.** Show that one arch of a cycloid generated by a circle of radius $R$ has length $8R$.
[§12.2 #13]

parametric equations:

$x=Rt-R\sin t,\ y=R-R\cos t$

$\Rightarrow x'=R-R\cos t,\ y'=R\sin t$

We get $x'(t)^2+y'(t)^2=(R-R\cos t)^2+(R\sin t)^2$

$=R^2-2R^2\cos t+R^2\cos^2 t+R^2\sin^2 t$

$=2R^2-2R^2\cos t=2R^2(1-\cos t)$

$=4R^2\sin^2\frac{t}{2}$

One arch of the cycloid is traced at t varies from 0 to $2\pi$.

$S=\int_0^{2\pi}\sqrt{x'(t)^2+y'(t)^2}\,dt=\int_0^{2\pi}\sqrt{4R^2\sin^2\frac{t}{2}}\,dt$

$=2R\int_0^{2\pi}\sin\frac{t}{2}\,dt=4R\int_0^{\pi}\sin u\,du$

$=-4R\cos u\Big|_0^{\pi}=-4R(-1-1)=8R$ ※

**8.** Find the minimum speed of a particle with trajectory $c(t) = (t^3 - 4t, t^2 + 1)$ for $t \ge 0$.
[§12.2 #21]

$x(t)=t^3-4t,\ y(t)=t^2+1$

$\Rightarrow x'(t)=3t^2-4,\ y'(t)=2t$

The speed is $\frac{ds}{dt}=\sqrt{(3t^2-4)^2+(2t)^2}$

$=\sqrt{9t^4-24t^2+16+4t^2}$

$=\sqrt{9t^4-20t^2+16}$

The square root function is an increasing function, hence the minimum speed occurs at the value of t where the function $f(t)=9t^4-20t^2+16$ has minimum value.

$f'(t)=36t^3-40t=4t(9t^2-10)=0,\ t=0$ or $\sqrt{\frac{10}{9}}$

$f(0)=16,\ f(\sqrt{\frac{10}{9}})=\frac{44}{9},\ \lim_{t\to\infty}f(t)=\infty$

$\therefore$ minimum value when $t=\sqrt{\frac{10}{9}}$

$\Rightarrow$ minimum speed is $\frac{ds}{dt}\Big|_{t=\sqrt{\frac{10}{9}}}=\sqrt{\frac{44}{9}}\approx 2.21$ ※

1. a. For the endpoint $x=1$, the series becomes $\sum_{n=2}^{\infty} \frac{1}{\ln n}$,

$\because n > \ln n$, $\frac{1}{n} < \frac{1}{\ln n}$, $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges

$\therefore$ By Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

For the endpoint $x=-1$, the series becomes $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$

With $a_n = \frac{1}{\ln n}$, $a_1 \geq a_2 \geq \cdots \geq 0$ and $\lim_{n \to \infty} a_n = 0$

By Leibniz Test, $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ converges.

b. For the endpoint $x = 2+e^{-1}$, the series becomes $\sum_{n=12}^{\infty} 1$

$\because a_n = 1$, $\{a_n\}$ does not converge to zero

$\therefore$ By the Divergence Test, $\sum_{n=12}^{\infty} 1$ diverges.

For the endpoint $x = 2-e^{-1}$, the series becomes $\sum_{n=12}^{\infty} (-1)^n$

$\because a_n = (-1)^n$, $\{a_n\}$ does not converge to zero.

$\therefore$ By the Divergence Test, $\sum_{n=12}^{\infty} (-1)^n$ diverges.