Calculus Homework Assignment 6

Class: ____________________

Student Number: ____________________

Name: 吴彦霖

1. Let $\mathbf{r}(t) = \langle t^2, 1-t, 4t \rangle$. Calculate the derivative of $\mathbf{r}(t) \cdot \mathbf{a}(t)$ at $t = 2$, assuming that $\mathbf{a}(2) = \langle 1, 3, 3 \rangle$ and $\mathbf{a}'(2) = \langle -1, 4, 1 \rangle$.
[§14.2 #31]

$\frac{d}{dt} r(t) \cdot a(t) = r'(t) \cdot a(t) + r(t) \cdot a'(t)$ &

$r'(t) = \langle 2t, -1, 4 \rangle$.

Hence, $\left.\frac{d}{dt} r(t) \cdot a(t)\right|_{t=2}$

$= r'(2) \cdot a(2) + r(2) \cdot a'(2)$

$= \langle 4, -1, 4 \rangle \cdot \langle 1, 3, 3 \rangle + \langle 4, -1, 8 \rangle \cdot \langle -1, 4, 1 \rangle$

$= 13$.

2. Find the general solution $\mathbf{r}(t)$ of the differential equation and the solution with the given initial condition.

$$\frac{d\mathbf{r}}{dt} = \langle e^{2t}, e^{t}, e^{-2t} \rangle, \quad \mathbf{r}(0) = \langle 4, -2, 3 \rangle$$

[§14.2 #50]

$r(t) = \int \langle e^{2t}, e^{t}, e^{-2t} \rangle \, dt$

$= \langle \frac{1}{2} e^{2t}, e^{t}, -\frac{1}{2} e^{-2t} \rangle + \underset{\text{constant}}{C}$.

$\because r(0) = \langle \frac{1}{2} e^{0}, e^{0}, -\frac{1}{2} e^{0} \rangle + C = \langle 4, -2, 3 \rangle$

$\therefore C = \langle 4, -2, 3 \rangle - \langle \frac{1}{2}, 1, -\frac{1}{2} \rangle$

$= \langle \frac{7}{2}, -3, \frac{7}{2} \rangle$

Hence, $r(t) = \langle \frac{1}{2} e^{2t} + \frac{7}{2}, e^{t} - 3, -\frac{1}{2} e^{-2t} + \frac{7}{2} \rangle$.

3. Which of the following is an arc length parametrization of a circle od radius 4 centered at the origin? Please state your reason!

a. $\mathbf{r}_1(t) = \langle 4\sin t, 4\cos t \rangle$

b. $\mathbf{r}_2(t) = \langle 4\sin 4t, 4\cos 4t \rangle$

√**c.** $\mathbf{r}_3(t) = \langle 4\sin \frac{t}{4}, 4\cos \frac{t}{4} \rangle$ [§14.3 #14]

Check: $\|r'(t)\| = 1$, $\forall t$.

a.

$\|r_1'(t)\| = \sqrt{(4\cos t)^2 + (-4\sin t)^2} = 4 \neq 1$.

b.

$\|r_2'(t)\| = \sqrt{(16\cos 4t)^2 + (-16\sin 4t)^2} = 16 \neq 1$.

c.

$\|r_3'(t)\| = \sqrt{(\cos \frac{t}{4})^2 + (\sin \frac{t}{4})^2} = 1$.

Hence, r_3 is the arc length parametrization of the circle.

4. Let $\mathbf{r}(t) = \langle 2t, 1-2t, t \rangle$.

a. Calculate $s(t) = \int_0^t \|\mathbf{r}'(u)\| \, du$ as a function of t.

b. Find the inverse $\varphi(s) = t(s)$ and show that $\mathbf{r}_1(s) = \mathbf{r}(\varphi(s))$ is an arc length parametrization. [like §14.3 #15]

a. $\|r'(t)\| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$.

$\therefore s(t) = \int_0^t 3 \, du = 3t$.

b. Finding $\varphi(s) = t(s)$ by solving $s = 3t, \forall t$.

$\Rightarrow t = \varphi(s) = \frac{s}{3}$, i.e., $\varphi(s) = s/3$.

$\Rightarrow r_1(s) = r(s/3) = \langle \frac{2s}{3}, 1 - \frac{2s}{3}, \frac{s}{3} \rangle$

Now, we check $\|r_1'(s)\| = 1$.

$\because r_1'(s) = \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$

$\therefore \|r_1'(s)\| = 1$

Hence, $r_1(s)$ is an arc length parametrization.

(Over Please)

5. Calculate $\mathbf{r}'(t)$, $\mathbf{T}(t)$, and $k(t)$. $K(t) = \frac{\|r' \times r''\|}{\|r'\|^3}$

$$\mathbf{r}(t) = \langle 4\cos t, t, 4\sin t \rangle$$

[like §14.4 #9]

$r'(t) = \langle -4\sin t, 1, 4\cos t \rangle \Rightarrow \|r'(t)\| = \sqrt{17}$

$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{17}} \langle -4\sin t, 1, 4\cos t \rangle.$

$r''(t) = \langle -4\cos t, 0, -4\sin t \rangle$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -4\sin t & 1 & 4\cos t \\ -4\cos t & 0 & -4\sin t \end{vmatrix}$$

$= -4\sin t\, i - 16\, j + 4\cos t\, k$

$= 4 \langle -\sin t, -4, \cos t \rangle$

$\Rightarrow \|r'(t) \times r''(t)\| = 4\sqrt{17}.$

Thus, $K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{4}{17}.$

6. Find $\mathbf{N}$ at the point indicated.

$$\langle t^{-1}, t, t^2 \rangle, \quad t = -1$$

Use $N(t) = \frac{r''(t) - v'(t)T(t)}{\|r''(t) - v'(t)T(t)\|}$ [§14.4 #45]

where $v(t) = \|r'(t)\|$ (P.751, Eq.(10)).

Let $r(t) = \langle t^{-1}, t, t^2 \rangle$

$\Rightarrow r'(t) = \langle -t^{-2}, 1, 2t \rangle$, $\|r'(t)\| = \sqrt{t^{-4} + 1 + 4t^2}$

$r''(t) = \langle 2t^{-3}, 0, 2 \rangle$

$v(t) = \sqrt{t^{-4} + 1 + 4t^2}$, $v'(t) = \frac{-2t^{-5} + 4t}{\sqrt{t^{-4} + 1 + 4t^2}}$

$\Rightarrow r'(-1) = \langle -1, 1, -2 \rangle$, $\|r'(-1)\| = \sqrt{6}$

$r''(-1) = \langle -2, 0, 2 \rangle$

$T(-1) = \frac{r'(-1)}{\|r'(-1)\|} = \frac{1}{\sqrt{6}} \langle -1, 1, -2 \rangle$

$\Rightarrow r''(-1) - v'(-1)T(-1) = \frac{1}{3} \langle -7, 1, 4 \rangle$

$\|r''(-1) - v'(-1)T(-1)\| = \frac{\sqrt{66}}{3}.$

Hence, $N(-1) = \frac{1}{\sqrt{66}} \langle -7, 1, 4 \rangle.$

7. Find the decomposition of $\mathbf{a}(t)$ into tangential and normal components at the point indicated, as in Example 6 (Page 758).

$$\mathbf{r}(t) = \langle t, e^t, te^t \rangle, \quad t = 0$$

$$a(t) = a_T(t)\, T(t) + a_N(t)\, N(t).$$

[§14.5 #35]

$V(t) = r'(t) = \langle 1, e^t, (1+t)e^t \rangle$

$\Rightarrow V(0) = \langle 1, 1, 1 \rangle$, $\|V(0)\| = \sqrt{3}.$

$a(t) = r''(t) = \langle 0, e^t, (2+t)e^t \rangle$

$\Rightarrow a(0) = \langle 0, 1, 2 \rangle$

$T(0) = \frac{V(0)}{\|V(0)\|} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$

$a_T(0) = a(0) \cdot T(0) = \langle 0, 1, 2 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle = \sqrt{3}.$

$a_N(0)\, N(0) = a(0) - a_T(0)\, T(0)$

$= \langle 0, 1, 2 \rangle - \sqrt{3} \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle = \langle -1, 0, 1 \rangle$

$\because$ N is a unit vector and $a_N \geq 0$, we have

$a_N(0) = \|a_N(0)\, N(0)\| = \sqrt{2}$ and

$N(0) = \frac{a_N(0) N(0)}{a_N(0)} = \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle$

$a(0) = a_T(0)\, T(0) + a_N(0)\, N(0) = \sqrt{3} \cdot T(0) + \sqrt{2}\, N(0),$

where $T(0) = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$, $N(0) = \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle.$

8. At time t_0, a moving particle has velocity vector $\mathbf{v} = 2\mathbf{i}$ and acceleration vector $\mathbf{a} = 3\mathbf{i} + 18\mathbf{k}$. Determine the curvature $k(t_0)$ of the particle's path at time t_0. [§14.5 #44]

$\because$ $r'(t_0) = V = 2i$ and

$r''(t_0) = a = 3i + 18k$

$$\Rightarrow r'(t_0) \times r''(t_0) = \begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 3 & 0 & 18 \end{vmatrix} = -36\, j,$$

$\|r'(t_0) \times r''(t_0)\| = 36,$

$\|r'(t_0)\| = 2.$

$\therefore k(t_0) = \frac{\|r'(t_0) \times r''(t_0)\|}{\|r'(t_0)\|^3} = \frac{9}{2}.$