Calculus Homework Assignment 8

Class: ______________________

Student Number: ______________________

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1. Determine the global extreme values of the function $f(x,y) = e^{-x^2-y^2}$ on the given set $x^2+y^2 \le 4$ without using calculus. [like §15.7 #29]

$0 \le x^2+y^2 \le 4$

$\Rightarrow 0 \ge -(x^2+y^2) \ge -4$

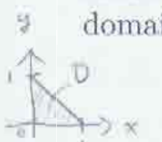
$\Rightarrow e^0 \ge e^{-(x^2+y^2)} \ge e^{-4}$

$\Rightarrow 1 \ge f(x,y) \ge e^{-4}$

$\Rightarrow$ Maximum $= 1$

Minimum $= e^{-4}$

2. Determine the global extreme values of the function $f(x,y) = x^3 + x^2y + 2y^2$ on the given domain, $x, y \ge 0, x+y \le 1$. [§15.7 #39]



$f_x = 3x^2+2xy$

$f_y = x^2+4y$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow$ critical point $(0,0)$, $(6,-9)$ Not in D.

$f(0,0) = 0$.

Boundary

$0 \le x \le 1$, $y = 0 \Rightarrow f(x,y) = x^3$

Max: $f(1,0) = 1$

min: $f(0,0) = 0$

$0 \le y \le 1$, $x = 0 \Rightarrow f(x,y) = 2y^2$

Max: $f(0,1) = 2$

min: $f(0,0) = 0$

$x+y=1$ $(y = 1-x)$

$f(x,y) = x^3 + x^2(1-x) + 2(1-x)^2$

$= 3x^2 - 4x + 2$

$= 3(x-\frac{2}{3})^2 + \frac{2}{3}$

Max: $f(0,1) = 2$

min: $f(\frac{2}{3}, \frac{1}{3}) = \frac{2}{3}$

$\therefore$ global max $= 2$, when $(x,y) = (0,1)$

global min $= 0$, when $(x,y) = (0,0)$.

3. Find the minimum and maximum values of the function $f(x,y) = x^2y^4$ subject to the given constraint, $x^2 + 2y^2 = 6$. [§15.8 #10]

Let $g(x,y) = x^2+2y^2-6$.

$\nabla f = (2xy^4, 4x^2y^3)$

$\nabla g = (2x, 4y)$

$\Rightarrow \begin{cases} 2xy^4 = \lambda 2x \\ 4x^2y^3 = \lambda 4y \end{cases} \Rightarrow x^2 = y^2 \Rightarrow x = \pm y$ or $x=0$ or $y=0$

Solve x, y by the constraint:

If $x = y$, $g(x,y) = 3x^2-6 = 0 \Rightarrow x = \pm\sqrt{2}$, $y = \pm\sqrt{2}$

If $x = -y$, $g(x,y) = 3x^2-6 = 0 \Rightarrow x = \pm\sqrt{2}$, $y = \mp\sqrt{2}$

$f(\sqrt{2},\sqrt{2}) = f(\sqrt{2},-\sqrt{2}) = f(-\sqrt{2},\sqrt{2}) = f(-\sqrt{2},-\sqrt{2}) = 8$

When $x=0$ or $y=0$, $f(x,y) = 0$

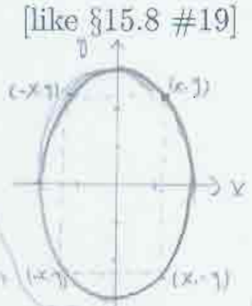
Hence minimum of f is 0

maximum of f is 8

4. Use Lagrange multipliers to find the maximum area of a rectangle inscribed in the ellipse (See the Figure 12 in Page 849):

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

[like §15.8 #19]



$f(x,y) = 4xy$ $(x>0, y>0)$

$g(x,y) = \frac{x^2}{4} + \frac{y^2}{9} - 1$

$\nabla f = (4y, 4x)$

$\nabla g = (\frac{x}{2}, \frac{2y}{9})$

$\therefore \begin{cases} 4y = \lambda \frac{x}{2} \\ 4x = \lambda \cdot \frac{2y}{9} \end{cases} \Rightarrow \frac{8y}{x} = \frac{18x}{y} \Rightarrow y = \pm\frac{3}{2}x$

$g(x, \frac{3}{2}x) = 0 \Rightarrow x = \sqrt{2}$, $y = \frac{3}{2}\sqrt{2}$

The maximum area $= 4 \cdot \sqrt{2} \cdot \frac{3}{2}\sqrt{2} = 12$

(Over Please)

5. Use symmetry to evaluate the double integral.

$$\iint_R (2 + x^2 y)\, dA, \quad R = [0,1] \times [-1,1]$$

[§16.1 #16]

Let $f(x,y) = x^2 y$

Then $f(x,-y) = -x^2 y = -f(x,y)$

$$\iint_R f(x,y)\, dA = \int_{-1}^{1}\int_0^1 f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 f(x,y)\, dA + \int_{-1}^{0}\int_0^1 f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 f(x,y)\, dA + \int_0^1\int_0^1 f(x,-y)\, dx\, dy$$

$$= \int_0^1\int_0^1 f(x,y)\, dA - \int_0^1\int_0^1 f(x,y)\, dx\, dy = 0.$$

Hence $\iint_R (2+x^2y)\, dA = \iint_R 2\, dA + \iint_R f(x,y)\, dA$

$$= 4$$

6. Evaluate

$$\int_0^1\int_0^1 \frac{y}{1+xy}\, dy\, dx.$$

[§16.1 #44]

$$\int_0^1\int_0^1 \frac{y}{1+xy}\, dy\, dx$$

$$= \int_0^1\int_0^1 \frac{y}{1+xy}\, dx\, dy$$

$$= \int_0^1 y \cdot \left[\frac{1}{y}\ln(1+xy)\right]\Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 \ln(1+y)\, dy$$

$$= \int_1^2 \ln u\, du \quad (u = 1+y,\ du = dy)$$

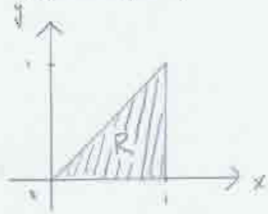
$$= u\ln u - u \Big|_{u=1}^{u=2}$$

$$= 2\ln 2 - 2 - (\ln 1 - 1)$$

$$= 2\ln 2 - 1$$

7. For the double integral, $\displaystyle\int_0^1\int_y^1 \frac{\sin x}{x}\, dx\, dy$, sketch the region. Then change the order of integration and evaluate. Explain the simplification achieved by interchanging the order.

[§16.2 #37]



$$\int_0^1\int_y^1 \frac{\sin x}{x}\, dx\, dy$$

$$= \int_0^1\int_0^x \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \frac{\sin x}{x} \cdot y\Big|_{y=0}^{y=x} dx$$

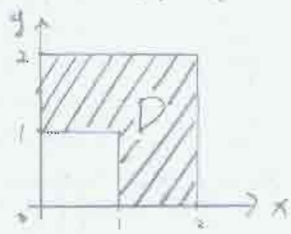
$$= \int_0^1 \sin x\, dx = -\cos x\Big|_{x=0}^{x=1} = 1 - \cos 1$$

When we compute $\int_0^1\int_y^1 \frac{\sin x}{x}\, dx\, dy$, we need to know the antiderivative for $\frac{\sin x}{x}$. If we change the order of integration, the integrand becomes $\sin x$, which was easier (than $\frac{\sin x}{x}$) to compute the antiderivative.

8. Sketch D where $0 \le x, y \le 2$ and x or y is greater than 1, and compute $\displaystyle\iint_D e^{x+y}\, dA$.

[§16.2 #46]

$$D = ([0,2]\times[0,2]) - ([0,1]\times[0,1])$$



$$\iint_D e^{x+y}\, dA = \int_0^2\int_0^2 e^{x+y}\, dx\, dy - \int_0^1\int_0^1 e^{x+y}\, dx\, dy$$

$$\left(\begin{array}{c}\text{Note that}\\ e^{x+y} = e^x e^y\end{array}\right) = \left(\int_0^2 e^x dx\right)\left(\int_0^2 e^y dy\right) - \left(\int_0^1 e^x dx\right)\left(\int_0^1 e^y dy\right)$$

$$= \left(\int_0^2 e^x dx\right)^2 - \left(\int_0^1 e^x dx\right)^2$$

$$= (e^2 - e^0)^2 - (e^1 - e^0)^2$$

$$= e^4 - 3e^2 + 2e$$