Calculus Homework Assignment 9

Class: 黃桐南

Student Number:

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1. Evaluate $\int\int\int_W f(x,y,z)\,dV$ for the function

$$f(x,y,y) = x + y$$

and region

$$W : y \le z \le x, 0 \le y \le x, 0 \le x \le 1$$

specified. [§16.3 #11]

2. Find the average of $f(x,y,z) = x^2 + y^2 + z^2$ over the region bounded by the planes $y + 2z = 1$, $x = 0$, $x = 1$, $z = 0$, and $y = 0$. [like §16.3 #33]

3. Sketch the region of integration and evaluate by changing to polar coordinates.

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (x^2 + y^2)\,dydx$$

[§16.4 #13]

4. Calculate the integral over the given region by changing to polar coordinates.

$$f(x,y) = |xy|, \qquad x^2 + y^2 \le 1$$

[§16.4 #23]

(Over Please)

5. Let $D = \Phi(R)$, where $\Phi(u,v) = (u^2, 2u+v)$ and $R = [1,2] \times [0,6]$. Calculate $\int\int_D y\, dA$.
[§16.5 #29]

6. Find an appropriate change of variables to evaluate

$$\int\int_R (x+y)^2 e^{x^2-y^2}\; dxdy$$

where R is the square with vertices

$$(1,0),\quad (0,1),\quad (-1,0),\quad (0,-1).$$

[§16.5 #38]

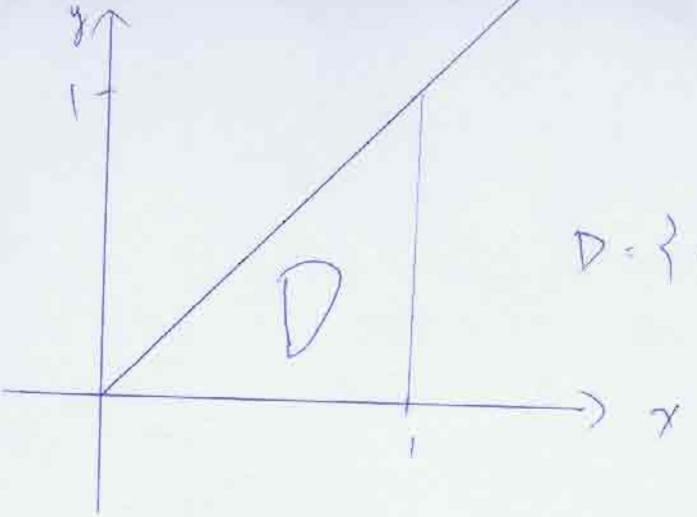
7. Compute the line integral of the vector field over the oriented curve.
$\mathbf{F} = \langle 4, y\rangle$, quarter circle $x^2+y^2 = 4$ with $x, y \leq 0$ oriented counterclockwise [§17.2 #24]

8. Let $\mathbf{F}$ be the **vortex vector field**.

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

Compute $\int_{C_R} \mathbf{F} \cdot ds$, where C_R is the circle of radius R centered at the origin oriented counterclockwise. Show that the result is independent of R. [§17.2 #42]

1.



$D = \{ (x,y) : 0 \le y \le x ,\ 0 \le x \le 1 \}$

$$\iiint_W f(x,y,z)\,dV = \iint_D \left(\int_y^x (x+y)\,dz \right) dA$$

$$= \iint_D (x+y)z \Big|_{z=y}^{z=x} dA$$

$$= \iint_D (x+y)(x-y)\,dA$$

$$= \iint_D (x^2-y^2)\,dA$$

$$= \int_0^1 \int_0^x (x^2-y^2)\,dy\,dx$$

$$= \int_0^1 x^2y - \frac{y^3}{3} \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{2}{3}x^3\,dx$$

$$= \frac{2}{12}x^4 \Big|_{x=0}^{x=1} = \frac{1}{6}$$

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2. $\because$ The region is bounded by the planes $y+2z=1$, $x=0$, $x=1$, $z=0$, $y=0$

$\Rightarrow$ let $W = \{(x,y,z) : 0 \le z \le 1-2y,\ 0 \le y \le \frac{1}{2},\ 0 \le x \le 1\}$

$\Rightarrow V = \text{Volume}(W) = \int_0^1 \int_0^{\frac{1}{2}} \int_0^{1-2y} dz\, dy\, dx = \frac{1}{4}$

$\Rightarrow$ The average of f over W is

$$\bar{f} = \frac{1}{V} \iiint_W f(x,y,z)\, dV$$

$$= 4 \cdot \int_0^1 \int_0^{\frac{1}{2}} \int_0^{1-2y} (x^2+y^2+z^2)\, dz\, dy\, dx$$

$$= 4 \int_0^1 \int_0^{\frac{1}{2}} (x^2+y^2)z + \frac{z^3}{3} \Big|_{z=0}^{z=1-2y} dy\, dx$$

$$= 4 \int_0^1 \int_0^{\frac{1}{2}} \left(x^2 - 2x^2y + y^2 - 2y^3 + \frac{(1-2y)^3}{3}\right) dy\, dx$$

$$= 4 \int_0^1 x^2y - x^2y^2 + \frac{y^3}{3} - \frac{y^4}{2} - \frac{(1-2y)^4}{24} \Big|_{y=0}^{y=\frac{1}{2}} dx$$

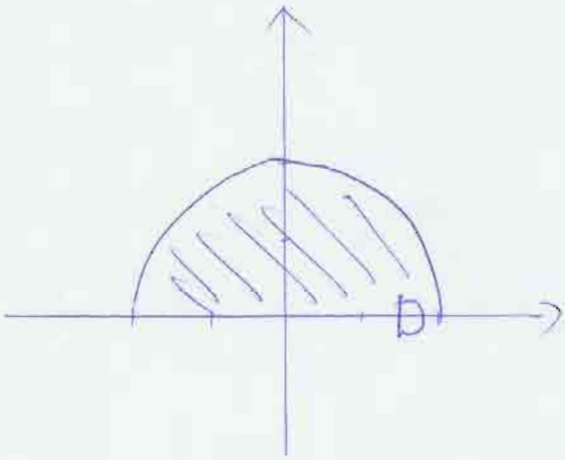
$$= 4 \int_0^1 \left(\frac{x^2}{4} + \frac{5}{96}\right) dx$$

$$= 4 \cdot \left(\frac{x^3}{12} + \frac{5x}{96} \Big|_{x=0}^{x=1} \right) = \frac{13}{24}$$ #

8. 13

let $D = \{ (x,y) : 0 \le y \le \sqrt{4-x^2},\ -2 \le x \le 2 \}$

$\Rightarrow D = \{ (x,y) : x^2+y^2 \le 4,\ 0 \le y \}$



let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow D = \{ (r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le \pi \}$

$\Rightarrow x^2+y^2 = r^2$

$$\Rightarrow \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx = \int_0^{\pi}\int_0^2 r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{r^4}{4}\Bigg|_{r=0}^{r=2} d\theta$$

$$= \int_0^{\pi} 4\,d\theta$$

$$= 4\theta\Bigg|_{\theta=0}^{\theta=\pi} = 4\pi$$

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4. let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow x^2+y^2 = r^2$

let $D = \{(x,y) : x^2+y^2 \le 1\} = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$\Rightarrow f(x,y) = |xy| = |r\cos\theta \cdot r\sin\theta| = \frac{1}{2}r^2|\sin 2\theta|$

$$\Rightarrow \iint_D f(x,y)\,dA = \int_0^{2\pi}\int_0^1 \frac{1}{2}r^2|\sin 2\theta| \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{8}|\sin 2\theta| \Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{8}|\sin 2\theta|\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{8}\sin 2\theta\,d\theta - \int_{\frac{\pi}{2}}^{\pi} \frac{1}{8}\sin 2\theta\,d\theta + \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{8}\sin 2\theta\,d\theta - \int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{8}\sin 2\theta\,d\theta$$

$$= \frac{-1}{16}\cos 2\theta \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} + \frac{1}{16}\cos 2\theta \Big|_{\theta=\frac{\pi}{2}}^{\theta=\pi} - \frac{1}{16}\cos 2\theta \Big|_{\pi}^{\frac{3\pi}{2}} + \frac{1}{16}\cos 2\theta \Big|_{\frac{3\pi}{2}}^{2\pi}$$

$$= \frac{2}{16} + \frac{2}{16} + \frac{2}{16} + \frac{2}{16} = \frac{1}{2}$$

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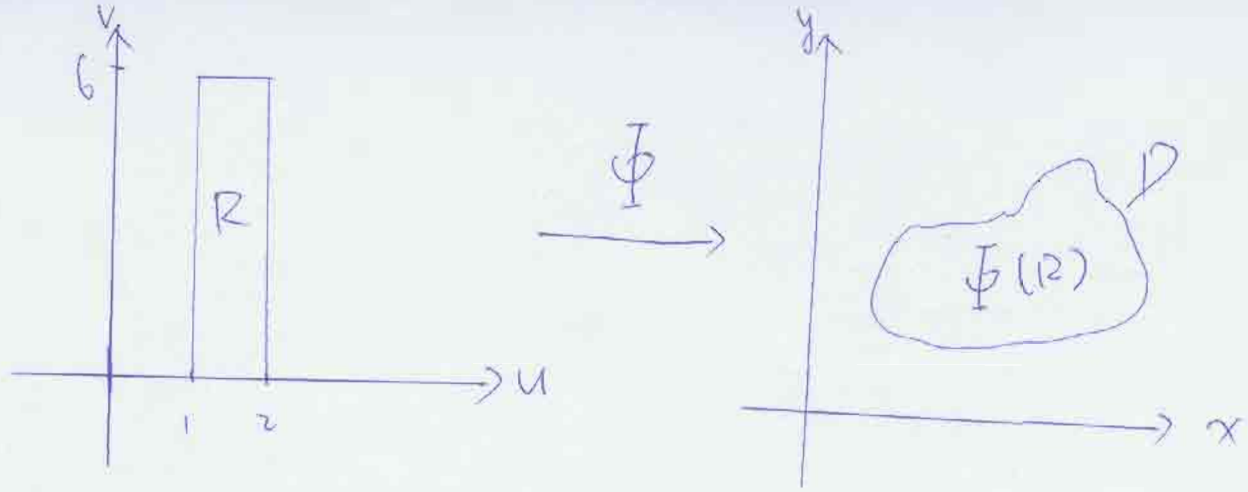
5.



By changing variables, $\iint_D y\,dA = \iint_R (2u+v)\cdot|J(\Phi)|\,du\,dv$

$\because x = u^2,\ y = 2u+v \Rightarrow J(\Phi) = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 2 & 1 \end{vmatrix} = 2u$

$$\Rightarrow \iint_D y\,dA = \int_0^6\int_1^2 (2u+v)\cdot 2u\,du\,dv$$

$$= \int_0^6\int_1^2 (4u^2+2uv)\,du\,dv$$

$$= \int_0^6 \frac{4}{3}u^3 + u^2 v\Big|_{u=1}^{u=2}\,dv$$

$$= \int_0^6 \left(3v+\frac{28}{3}\right)dv$$

$$= \frac{3}{2}v^2 + \frac{28}{3}v\Big|_{v=0}^{v=6} = 110$$

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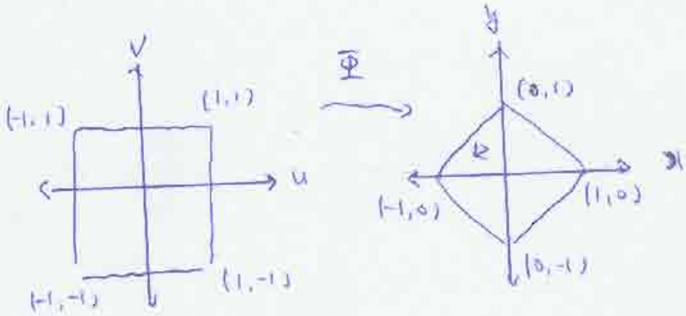
6

Let $\Phi \begin{cases} u = x + y \\ v = x - y \end{cases}$ or $\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$



$$\text{Jacobian } J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad |J| = \frac{1}{2}$$

$$\iint_R (x+y)^2 e^{x^2-y^2}\, dx\, dy = \int_{-1}^{1}\int_{-1}^{1} u^2 e^{uv} |J|\, dv\, du$$

$$= \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2} u^2 e^{uv}\, dv\, du$$

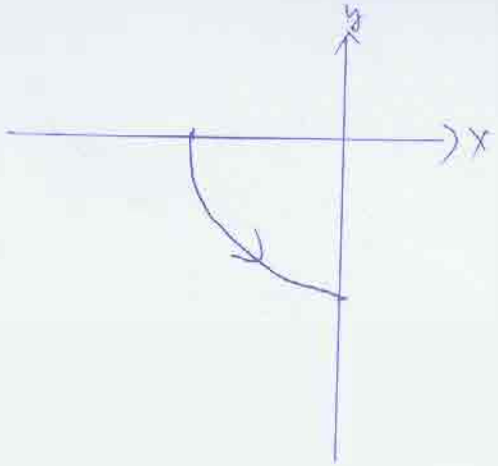
$$= \frac{1}{2}\int_{-1}^{1} \frac{1}{u} u^2 e^{uv}\Big|_{-1}^{1}\, du$$

$$= \frac{1}{2}\int_{-1}^{1} u\left[e^{u} - e^{-u}\right] du$$

$$= \frac{1}{2}\int_{-1}^{1} u e^{u}\, du - \frac{1}{2}\int_{-1}^{1} u e^{-u}\, du$$

$$= \frac{1}{2} e^{u}(u-1)\Big|_{-1}^{1} + \frac{1}{2} e^{-u}(u+1)\Big|_{-1}^{1} = 2e^{-1}$$

7.



The oriented counterclockwise path can be parametrized by

$$C(t) = \langle \cancel{2}\cos t, \cancel{4}\sin t \rangle, \quad \pi \le t \le \frac{3\pi}{2}$$

$$\Rightarrow F(c(t)) = \langle 4, \cancel{2}\sin t \rangle$$

$$\Rightarrow F(c(t)) \cdot c'(t) = \langle 4, \cancel{2}\sin t \rangle \cdot \langle -\cancel{2}\sin t, 2\cos t \rangle = -\cancel{8}\sin t + \cancel{\tfrac{1}{2}}\sin 2t$$

$$\Rightarrow \int_C F \cdot ds = \int_{\pi}^{\frac{3\pi}{2}} F(c(t)) \cdot c'(t)\, dt$$

$$= \int_{\pi}^{\frac{3\pi}{2}} \overset{-8\sin t + 2\sin 2t}{\left(\cancel{-4\sin t + \tfrac{1}{2}\sin 2t}\right)} dt$$

$$= \cancel{4\cos t - \frac{1}{4}\cos 2t} \Big|_{t=\pi}^{t=\frac{3\pi}{2}} \quad 2\left(4\cos t - \tfrac{1}{2}\sin 2t\right) = \cancel{\frac{9}{2}} \quad 10 \ \#$$

8. C_R can be parametrized by $c(t) = \langle R\cos t,\ R\sin t\rangle$, $0 \le t \le 2\pi$, $R > 0$

$\Rightarrow c'(t) = R\langle -\sin t, \cos t\rangle$ and $F(c(t)) = \frac{1}{R}\langle -\sin t, \cos t\rangle$

$\Rightarrow F(c(t))\cdot c'(t) = \frac{1}{R}\langle -\sin t, \cos t\rangle \cdot R\langle -\sin t, \cos t\rangle = 1$

$\therefore \int_C F\cdot ds = \int_0^{2\pi} F(c(t))\cdot c'(t)\,dt = \int_0^{2\pi} 1\cdot dt = 2\pi$ #

☆ $\int_c F\cdot ds$ ~~$= 2\pi$ always~~ no matter what value of R is always equals to 2π.