$\boxed{\text{Real Analysis I}}$ Examination II (12/03/2008)

Total Score : 100

(1) (15%)

(i) State the respectively following definition of the Lebesgue integral on $\mathbf{R}^d$: Simple functions; Bounded functions supported on a set of finite measure; Non-negative function; Integrable functions (the general case).

(ii) Prove that: The integral of Lebesgue integrable functions is linear, additive, monotonic, and satisfies the triangle inequality.

(iii) (Tchebychev inequality) Suppose $f \geq 0$, and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, prove that $m(E_\alpha) \leq \frac{1}{\alpha} \int f$.

(2) (20%) State and prove the following theorem respectively: Bounded convergence theorem; Fatou's lemma; Monotone convergence theorem; Lebesgue dominated convergence theorem.

(3) (20%) Prove that:

(i) The space $\mathbf{R}^d, d \geq 1$, is a complete vector space in its usual norm.

(ii) The space L^1 is a complete vector space in its metric.

(4) (i) (10%) Prove that if f is finite almost everywhere, then f is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty, \quad \text{if and only if} \quad \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

(ii) (10%) Let

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then prove that f is integrable on $\mathbf{R}^d$ if and only if $a > d$.

(5) (i) (5%) Prove that if f is uniformly continuous on $\mathbf{R}$ and integrable, then $\lim_{x\to\infty} f(x) = 0$.

(ii) (5%) Prove that if f is integrable on $\mathbf{R}^d$, real-valued and $\int_E f(x)\, dx \geq 0$ for every measurable E, then $f(x) \geq 0$ a.e x.

(iii) (5%) If f is integrable on $\mathbf{R}$, show that

$$F(x) = \int_{-\infty}^{x} f(t)\ dt$$

is uniformly continuous.

(6) (10%) Prove or disprove that if $f \in L^1(\mathbf{R}^d)$ and a sequence $\{f_n\} \subset L^1(\mathbf{R}^d)$ such that $\|f_n - f\|_{L^1} \to 0$, then $f_n(x) \to f(x)$ a.e. x.