Final exam of Ordinary Differential Equation II

2018.6.21

以下題目, 須詳列過程, 8題選5題作答。

1. Find the solution of the differential equation

$$y'' + y = sin(2t) \tag{1}$$

satisfying the initial condition

$$y(0) = 2, \quad y'(0) = 1 \tag{2}$$

2. Find the solution of the differential equation

$$2y'' + y' + 2y = g(t) \tag{3}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \leqslant t < 20 \\ 0, & 0 \leqslant t < 5 \ or \ t \geqslant 20 \end{cases} \tag{4}$$

Assume that the initial conditions are

$$y(0) = 0, \quad y'(0) = 0$$

This problem governs the charge on the capacitor in a simple electric circuit with a unit voltage pulse for $5 \leqslant t < 20$. Alternatively, y may represent the response of a damped oscillator subject to the applied force $g(t)$

3. (a) Solve the initial value problem:

$$X' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} X \tag{5}$$

satisfying the initial value condition

$$X(0) = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \tag{6}$$

(b) Describe the behavior of the solution as $t \to \infty$ and $t \to -\infty$

4. Find the solution of the following initial value problem

$$x' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix} \tag{7}$$

5. Consider the equation

$$2x(1+x)y'' + (3+x)y' - xy = 0 \tag{8}$$

   (a) Find the singular points of (8), and verify whether or not these singular points are regular singular points.
   (b) What is the indicial equation of (8) ?
   (c) Write the solutions of (8) in series form.

6. Use the method of undetermined coefficients to find a particular solution of

$$x' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = Ax + g(t) \tag{9}$$

7. (a) Find a particular solution of

$$y''' - 4y' = t + 3cost + e^{-2t}$$

   (b) Find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

$$y''' - y'' + y' - y = sect, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

8. Show that two solutions of the Legendre equation for $|x| < 1$ are

$$\begin{aligned} y_1(x) &= 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 + \\ &+ \sum_{m=3}^{\infty}(-1)^m \frac{\alpha...(\alpha-2m+2)(\alpha+1)...(\alpha+2m-1)}{(2m)!}x^{2m} \end{aligned}$$

$$\begin{aligned} y_2(x) &= x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 \\ &+ \sum_{m=3}^{\infty}(-1)^m \times \frac{(\alpha-1)...(\alpha-2m+1)(\alpha+2)...(\alpha+2m)}{(2m+1)!}x^{2m+1} \end{aligned}$$