Midterm of Ordinary Differential Equation II

2018.4.12

Choose eight problems to do from all ones. (10題選8題作答)

1. Find the solution of the given initial value problem. How does the solution behave as $t \to \infty$

$$y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0;\ y(0) = 1,\ y'(0) = -2,\ y''(0) = 0,\ y'''(0) = 3$$

2. (a) Find a particular solution of

$$y''' - 4y' = t + 3cost + e^{-2t}$$

   (b) Find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

$$y''' - y'' + y' - y = sect, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

3. (a) Find a series solution in powers of x of Airy's equation

$$y'' - xy = 0,\ -\infty < x < \infty$$

   (b) Find a solution of Airy's equation in powers of $x - 1$.

4. **The Hermite Equation**: The equation

$$y'' - 2xy' + \lambda y = 0, \quad -\infty < x < \infty$$

where λ is a constant, is known as the Hermite equation. It is an important equation in mathematical physics.

   (a) Find the frist four nonzero terms in each of two solutions about $x = 0$ and show that they form a fundamental set of solutions.

   (b) Observe that if λ is a nonnegative even integer, then one or the other of the seires solutions termnates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0, 2, 4, 6, 8$, and 10. Note that each polynomial is determined only up to a multiplicative constant.

   (c) The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x), H_1(x), ..., H_5(x)$

5. Show that two solutions of the Legendre equation for $|x| = 1$ are

$$\begin{aligned} y_1(x) &= 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 + \\ &+ \sum_{m=3}^{\infty}(-1)^m \frac{\alpha...(\alpha-2m+2)(\alpha+1)...(\alpha+2m-1)}{(2m)!}x^{2m} \end{aligned}$$

$$\begin{aligned} y_2(x) &= x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 \\ &+ \sum_{m=3}^{\infty}(-1)^m \times \frac{(\alpha-1)...(\alpha-2m+1)(\alpha+2)...(\alpha+2m)}{(2m+1)!}x^{2m+1} \end{aligned}$$

6. (a) Find all values of β for which all solutions of

$$x^2y'' + \beta y = 0$$

approach zero as $x \to 0$

(b) Finr γ so that the solution of the initial-value problem

$$x^2y'' - 2y = 0, \; y(1) = 1, \; y'(1) = \gamma$$

is bounded as $x \to 0$

7. Show that if $r_1, r_2, ..., r_n$ are all real and different, then $e^{r_1 t}, ..., e^{r_n t}$ are linearly independent on $-\infty < t < \infty$.

8. Find a formula involving integrals for a particular solution of the differential equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = g(x), \; x > 0$$

Hint: Verify that x, x^2, and x^3 are solutions of the homogeneous equation.

9. The Legendre polynomials play an important role in mathematical physics. For example, in solving Laplace's equation (the potential equation) in spherical coordinates, we encounter the equation

$$\frac{d^2F(\psi)}{d\psi^2} + cot\psi\frac{dF(\psi)}{\psi} + n(n+1)F(d\psi), \quad 0 < \psi < \pi$$

where n is a positive integer. Show that the change of variable $x = cos\phi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(arccosx)$

10. Determine the a_n so that the equation

$$\sum_{n=1}^{\infty} na_nx^{n-1} + 3\sum_{n=0}^{\infty} a_nx^n = 0$$

is satisfied. Try to identify the function represented by the series $\sum_{n=0}^{\infty} a_nx^n$