Ordinary Differential Equations Final Examination

Jan 16, 2015

Choose two problems from 1-4 and three problem from 5-8

1. Consider the following initial value problem

$$\begin{cases} x_1'(t) = r_1 x_1(t)\Big[1 - \dfrac{x_1(t)}{\kappa_1}\Big] - \alpha x_1(t)x_2(t) \ \text{ on } \ [0,\infty), \\ x_2'(t) = r_2 x_2(t)\Big[1 - \dfrac{x_2(t)}{\kappa_2}\Big] - \beta x_1(t)x_2(t) \ \text{ on } \ [0,\infty), \\ x_1(0) = x_{10} > 0, \ x_2(0) = x_{20} > 0, \end{cases} \qquad (I)$$

where α, β, r_1, r_2, κ_1 and κ_2 are positive constants. Let $X(t) = (x_1(t), x_2(t))$ be a solution of (I) on $[0,\infty)$. How about $\lim_{t\to\infty} X(t)$? Show your answer.

2. Consider the following initial value problem

$$\begin{cases} x'(t) = [A - By(t)]x(t) \ \text{ on } \ [0,\infty), \\ y'(t) = [Cx(t) - D]y(t) \ \text{ on } \ [0,\infty), \\ x(0) = x_0 > 0, \ y(0) = y_0 > 0, \end{cases} \qquad (II)$$

where A, B, C and D are positive constants. Let $X(t) = (x(t), y(t))$ be a solution of (II) on $[0,\infty)$. How about $\lim_{t\to\infty} X(t)$? Show your answer.

3. Consider the following initial value problem

$$\begin{cases} mx''(t) + cx'(t) + kx(t) = 0 \ \text{ on } \ [0,\infty), \\ x(0) = x_0 > 0, \ x'(0) = x_1 > 0, \end{cases} \qquad (III)$$

where m(mass), c(friction), and k(restoring force) are positive constants. Prove that every solution $x(t)$ of (III) satisfies $\lim_{t\to\infty} x(t) = 0 = \lim_{t\to\infty} x'(t)$.

4. Derive the Kepler's Equation, and show the Kepler's second law.

5. Find the solutions structure of the $(R - L - C)$ equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

with respect to the parameters R, L, C.

6. Consider the linear system $\begin{cases} X' = AX \\ X(0) = x_0 \end{cases}$. $A \in R^{n\times n} \Rightarrow X(t, x_0) = e^{At}x_0$

where $e^{At} = I + \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k$

Find $X(t, x_0)$ of the following problem (1)-(2) respectively.

(1)

$$A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix} \quad x_0 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

(2)

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix} \quad x_0 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

7. Consider the following predator-prey system

$$\begin{cases} x' = \gamma x(1 - \dfrac{x}{K}) - \alpha xy & , \\ y' = y(\beta x - d), & \gamma, K, \alpha, \beta > 0 \\ x(0) > 0, y(0) > 0. \end{cases}$$

Show that the solutions x(t),y(t) are defined for all $t > 0$ and the solutions are positive and bounded for all $t > 0$.

8. Consider the population model

$$\begin{cases} \dfrac{dx}{dt} = rx(1 - \dfrac{x}{k}) - \dfrac{mx}{a + x} y, \\ \dfrac{dy}{dt} = sy(1 - \dfrac{y}{hx}), \\ x(0) > 0, y(0) > 0. \end{cases}$$

Show that the solutions $x(t)$,$y(t)$ are positive and bounded.