Real Analysis I: Examples and Convergent Property of Lebesgue Integrable Functions

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1 Lebesgue integrable but not Riemann Integrable

Remark 1.1 We note that if a real-valued function f is Riemann integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$. But the reverse is not true always. The following is such an example.

Example 1.1 Let

$$f(x) = \begin{cases} 1 \text{ if x is an irrational number in } [0,1] \\ 0 \text{ otherwise,} \end{cases} \tag{1.1}$$

and

$$g(x) = \begin{cases} 0 \text{ if x is an irrational number in } [0,1] \\ 1 \text{ otherwise.} \end{cases} \tag{1.2}$$

Then f and g are all Lebesgue integrable but not Riemann integrable.

Proof. Let $P : 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ be any partition of $[0, 1]$. Then it is easy to see that $\sup_{x_{j-1} \le x \le x_j} f(x) = 1$ and $\inf_{x_{j-1} \le x \le x_j} f(x) = 0$ for all $j = 1, \cdots N$, and hence the upper and lower Riemann sums of f w.r.t. P satisfy

$$U(P, f) = 1 \text{ and } L(P, f) = 0.$$

Hence f is not Riemann integrable. Similarly, we also see that g is not Riemann integrable. Let $E_1 = \{x : x \text{ is a rational number of } [0, 1]\}$ and $E_2 = \{x : x \text{ is a irrational number of } [0, 1]\}$. Then f and g are all simple functions satisfying

$$\begin{aligned} f(x) &= 0 \cdot \chi_{E_1}(x) + 1 \cdot \chi_{E_2}(x) \\ g(x) &= 1 \cdot \chi_{E_1}(x) + 0 \cdot \chi_{E_2}(x), \end{aligned} \tag{1.3}$$

and thus, by the definition of the Lebesgue integral for simple functions, we obtain that f and g are Lebesgue integrable and

$$\begin{array}{c} \int_{[0,1]} f = 0 \cdot m(E_1) + 1 \cdot m(E_2) = 1 \\ \int_{[0,1]} g = 1 \cdot m(E_1) + 0 \cdot m(E_2) = 0. \end{array} \tag{1.4}$$

2 Necessary Conditions for Bounded Convergent Theorem

The following two examples illustrate the conditions of Bounded Convergent Theorem are sharp.

Example 2.1 (Necessary for Bounded Functions) For each positive integer n, let

$$f_n(x) = \begin{cases} n & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 0 & \text{if } x \in (0, 1] \\ \infty & \text{if } x = 0. \end{cases} \tag{2.1}$$

Then $\{f_n\}$ is a sequence of Lebesgue integrable with unbounded function on $[0, 1]$. Furthermore we easily have $f_n(x) \to f(x)$ as $n \to \infty$, and $\int_{[0,1]} f_n(x)dx = n \cdot m([0, \frac{1}{n}]) + 0 \cdot m([\frac{1}{n}, 1]) = 1$ for all n, and thus $1 = \int_{[0,1]} f_n(x)dx \not\to \int_{[0,1]} f(x)dx = 0$ as $n \to \infty$.

Example 2.2 (Necessary for Bounded Support) For each positive integer n, let

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in [0, n] \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Then for each n, $\int_{[0,\infty)} f_n = 1$ and $\text{supp}(f_n) = [0, n] \to [0, \infty)$ as $n \to \infty$. Furthermore we easily have $f_n \to f \equiv 0$ as $n \to \infty$, and thus $1 = \int_{[0,\infty)} f_n(x)dx \not\to \int_{[0,\infty)} f(x)dx = 0$ as $n \to \infty$.

3 Monotone Convergent Theorem and its Applications

First we state the Monotone Convergence Theorem (MCT) in the following.

Theorem 31. Suppose $\{f_n\}$ is a sequence of non-negative measurable function with $f_n \nearrow f$. Then

$$\lim_{n\to\infty} \int f_n = \int f. \tag{3.1}$$

Now, we give two applications of Monotone Convergence Theorem in the following

Example 3.1 (Borel-Cantelli lemma)

Suppose $\{E_n\}$ is a sequence of measurable sets satisfying

$$\sum_{n=1}^{\infty} m(E_n) < \infty. \tag{3.2}$$

Let

$$F = \{x : x \text{ belongs to infinitely many sets} E_n\}. \tag{3.3}$$

Then F has measure zero, i.e., $m(F) = 0$.

Proof. For each n, let

$$a_n(x) = \chi_{E_n}(x). \tag{3.4}$$

Then we note that $x \in F$ if and only if $\sum\limits_{n=1}^{\infty} a_n(x) = \infty$, and $\int a_n = m(E_n)$. By (3.2) and monotone convergent theorem, we have $\int \sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} m(E_n) < \infty$, and thus we obtain $m(F) = 0$. ■

Actually, the set F in (3.3) satisfies $F = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{n=k}^{\infty} E_n$, and from this we also can prove that $m(F) = 0$. See Exercise 16 of Chapter 1.

Example 3.2

Consider the function

$$f(x) = \begin{cases} \frac{1}{|x|^{d+1}} & \text{if } \ x \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Then f is integrable outside any ball, $|x| \geq \epsilon$, and

$$\int_{|x|\geq\epsilon} f(x) \leq \frac{C}{\epsilon}, \quad \text{for some constant} \quad C > 0. \tag{3.6}$$