Real analysis midterm I

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1 Let $\{x_n\}$ be a sequence of strictly positive real numbers.

(1.a) State the following definitions respectively by $\epsilon - N$ method:
A cluster point; lim sup x_n; lim inf x_n.

(1.b) Show that lim sup x_n and lim inf x_n is the largest and smallest cluster points of the sequence $\{x_n\}$ respectively.

(1.c) If $\{x_n\}$ is a sequence of real numbers, then show that

$$lim\ inf x_n = lim\ sup x_n = l$$

if and only if l is the limit of $\{x_n\}$

(1.d) Show that

$$lim\ inf\ \frac{x_{n+1}}{x_n} \leq lim\ inf\ x_n^{1/n} \leq lim\ sup x_n^{1/n} \leq lim\ sup\ \frac{x_{n+1}}{x_n}$$

(Show your work by the $\epsilon - N$ definition of lim sup and lim inf)

2 Show that there exist disjoint sets E_1, E_2,...,E_k,...such that

$$|\bigcup E_k|_e < \sum |E_k|_e$$

with strict inequality.

3 State the following definition respectively: (a) Outer Measure (b) The Cantor Set (c) Lebesgue Measurable Set. Let

$$S = \{E \subset R^n | E \text{ is a measurable set}\}.$$

Prove that S is a σ-algebra.

4 State the following definition: the Cantor Set C of the interval $[0, 1]$; Borel set. Prove that (a) the Cantor set C is uncountable, perfect, measurable set, and find the measure of C; every Borel set is measurable.

5 Let $\{A\}$ be a collection of sets A. State the definition respectively: A_σ set; G_δ set. Let $E \subset R^d$. Prove the following statements are equivalent. (a) E is measurable. (b) E^c is measurable. (c) For every set A, $m(A) = m(A \bigcap E) + m(A - E)$. (d) Given $\varepsilon > 0$, there is a closed set F such that $F \subset E$ and $m^*(E - F) < \varepsilon$. (e)There is a G_δ set with $E \subset G_\delta$, $m^*(G_\delta - E) = 0$. (f)There is a F_σ set with $F_\sigma \subset E$, $m^*(E - F_\sigma) = 0$. If $m^*(E)$ is finite, then the above statements are equivalent the following:
(g)Given $\varepsilon > 0$, there is a union U of open intervals such that

$$m^*(U \Delta E) < \varepsilon.$$

6 Construct a Nonmeasurable Set.