Solutions of ODEs Final Examination

January 13, 2014

1. (i) Consider the following initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(t, x), \\ x(t_0) = x_0, \end{cases} \tag{IVP}$$

where $f : D \subseteq \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ and D is an open set of $\mathbf{R} \times \mathbf{R}^n$ containing (t_0, x_0). State the local existence-uniqueness theorem of (IVP).

(ii) Show that if $f \in C(\mathbf{R})$ and $g \in C^1(\mathbf{R})$, then the following initial value problem

$$\begin{cases} y'' + f(y)y' + g(y) = 0, \\ y(t_0) = a, \quad y'(t_0) = b, \end{cases}$$

has a unique solution locally.

Solution.

(i) If $f(t, x) \in C^1(D)$, then there exists an $c > 0$ such that (IVP) has a unique solution on $[t_0 - c, t_0 + c]$.

(ii) Let $x_1(t) = y(t)$ and $x_2(t) = y'(t) + \int_a^y f(s)\, ds$. Then we have the following first order linear equation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_2 - \displaystyle\int_a^{x_1} f(s)\, ds \\ -g(x_1) \end{bmatrix} \equiv F(x_1, x_2) \in C^1(\mathbf{R}, \mathbf{R}). \tag{$*$}$$

By the local existence-uniqueness theorem, $(*)$ has a unique solution locally. Therefore, the initial value problem has a unique solution locally.

3. Consider the following initial value problem

$$\begin{cases} X' = AX = \begin{bmatrix} -3 & 3/4 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\ X(0) \neq \mathbf{0}. \end{cases} \tag{LC_*}$$

Find a fundamental matrix of (LC_*) and describe the behavior of the solution as $t \to \infty$.

<u>**Solution.**</u>

First, we have that $v_1 = [3 \;\; 10]^T$, $v_2 = [1 \;\; 2]^T$ are the eigenvectors of A corresponding to the eigenvalues $\lambda_1 = -1/2$, $\lambda_2 = -3/2$ respectively. Then the general solution of (LC_*) is

$$X(t) = c_1 e^{-t/2} \begin{bmatrix} 3 \\ 10 \end{bmatrix} + c_2 e^{-3t/2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{where } c_1, c_2 \in \mathbf{R}.$$

Hence, a fundamental matrix of (LC_*) is given by

$$\Phi(t) = \begin{bmatrix} 3e^{-t/2} & e^{-3t/2} \\ 10e^{-t/2} & 2e^{-3t/2} \end{bmatrix}.$$

Furthermore, we obtain $\lim_{t \to \infty} X(t) = 0$.

5. Consider the following linear periodic system

$$X' = A(t)X, \tag{LP}$$

where $A(t) = [a_{ij}(t)]_{n\times n} \in \mathbf{R}^{n\times n}$ is continuous on $\mathbf{R}$ and $A(t) = A(t+T)$.

(i) State the Floquet Theorem, the definition of the Floquet multipliers (characteristic multipliers) of (LP).

(ii) Explain that the Floquet multipliers (characteristic multipliers) of (LP) are uniquely determined by the system (LP).

(iii) Let

$$A(t) = \begin{bmatrix} 1 & 1 \\ 0 & \dfrac{\sin t + \cos t}{2 + \sin t - \cos t} \end{bmatrix} = A(t+2\pi).$$

Compute the Floquet multipliers (characteristic multipliers) of (LP).

Solution.

(i) Floquet Theorem: If $\Phi(t)$ is a fundamental matrix of (LP), then so is $\Phi(t+T)$. Moreover, there exist $R \in \mathbf{C}^{n\times n}$ and nonsingular $P(t) \in \mathbf{C}^{n\times n}$ with $P(t) = P(t+T)$ such that $\Phi(t) = P(t)e^{tR}$.
Floquet multipliers: The eigenvalues of nonsingular matrix C with $\Phi(t+T) = \Phi(t)C$ are called the Floquet multipliers of (LP).

(ii) Let $\Phi(t)$ and $\Phi_*(t)$ be two fundamental matrices of (LP). Then, there exists a nonsingular $P \in \mathbf{C}^{n\times n}$ such that $\Phi(t) = \Phi_*(t)P$. On the other hand, by the Floquet theorem, we have $\Phi(t+T) = \Phi(t)C_1$ and $\Phi_*(t+T) = \Phi_*(t)C_2$ for some nonsingular matrices C_1, C_2. Moreover, there exist R_1, $R_2 \in \mathbf{C}^{n\times n}$ such that $C_1 = e^{TR_1}$ and $C_2 = e^{TR_2}$, which imply

$$\Phi(t+T) = \Phi(t)e^{TR_1} \quad \text{and} \quad \Phi_*(t+T) = \Phi_*(t)e^{TR_2}.$$

Then, we have

$$\Phi_*(t+T)P = \Phi(t+T) = \Phi(t)e^{TR_1} = \Phi_*(t)Pe^{TR_1},$$

that is, $\Phi_*(t+T) = \Phi_*(t)Pe^{TR_1}P^{-1}$. Hence, C_1 and C_2 are similar, which implies that C_1 and C_2 have the same eigenvalues.

(iii) First, it is easy to see that $X_1(t) = [\,e^t \;\; 0\,]^T$ is a solution of (LP). From (LP), we have the following first order equations

$$x_1'(t) = x_1(t) + x_2(t), \tag{$*_1$}$$

and

$$x_2'(t) = \frac{\sin t + \cos t}{2 + \sin t - \cos t} x_2(t). \tag{$*_2$}$$

Equation $(*_2)$ is separable and then we obtain $x_2(t) = 2 + \sin t - \cos t \neq 0$. Using the method of variation of parameters, we substitute $x_1(t) = u(t)e^t$ in $(*_1)$ and then $u(t)$ satisfies

$$u'(t) = (2 + \sin t - \cos t)e^{-t}.$$

So we get $u(t) = (-2 - \sin t)e^{-t}$ and $x_1(t) = -2 - \sin t$. Then $X_2(t) = \begin{bmatrix} -2 - \sin t & 2 + \sin t - \cos t \end{bmatrix}^T$ is also a solution of (LP). Moreover, $X_1(t)$ and $X_2(t)$ are linearly independent by the Wronskian $W(X_1, X_2) \neq 0$ for all $t \in \mathbf{R}$. Now, we let

$$\Phi(t) = [X_1(t) \;\; X_2(t)] = \begin{bmatrix} e^t & -2 - \sin t \\ 0 & 2 + \sin t - \cos t \end{bmatrix},$$

then $\Phi(t)$ is a fundamental matrix of (LP).

By the Floquet Theorem, $\Phi(t + 2\pi)$ is also a fundamental matrix and there exists a nonsingular 2×2 matrix C such that

$$\Phi(t + 2\pi) = \Phi(t)C, \quad \text{for all} \quad t \in \mathbf{R},$$

which imply

$$C = \Phi^{-1}(0)\Phi(2\pi) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2\pi} & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2\pi} & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the Floquet multipliers are $e^{2\pi}$, 1.