Ordinary Differential Equations Final Examination

January 13, 2014

Instruction: Please choose four from all five questions to answer.

1. **(i)** Consider the following initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \\ x(t_0) = x_0, \end{cases} \tag{IVP}$$

where $f : D \subseteq \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ and D is an open set of $\mathbf{R} \times \mathbf{R}^n$ containing (t_0, x_0). State the local existence-uniqueness theorem of (IVP).

(ii) Show that if $f \in C(\mathbf{R})$ and $g \in C^1(\mathbf{R})$, then the following initial value problem

$$\begin{cases} y'' + f(y)y' + g(y) = 0, \\ y(t_0) = a, \quad y'(t_0) = b, \end{cases}$$

has a unique solution locally.

2. Consider the following initial value problem

$$X' = AX, \quad X(0) = X_0, \tag{LC}$$

where $A \in \mathbf{R}^{n\times n}$ and $X_0 \in \mathbf{R}^n$. Show that

(i) if all of the eigenvalues of A have negative real parts, then for all $X_0 \in \mathbf{R}^n$

$$\lim_{t\to\infty} X(t) = \mathbf{0},$$

where $X(t)$ is the solution of (LC).

(ii) if A has an eigenvalue $\lambda = a + ib$ with $Re(\lambda) > 0$, then there exists an $X_0 \in \mathbf{R}^n$, $X_0 \neq \mathbf{0}$, such that

$$\lim_{t\to\infty} X(t) \neq \mathbf{0},$$

where $X(t)$ is the solution of (LC).

3. Consider the following initial value problem

$$\begin{cases} X' = AX = \begin{bmatrix} -3 & 3/4 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\ \\ X(0) \neq \mathbf{0}. \end{cases} \qquad (LC_*)$$

Find a fundamental matrix of (LC_*) and describe the behavior of the solution as $t \to \infty$.

4. Consider the following two-dimensional linear system

$$X' = AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad (LC_{**})$$

where a, b, c, d are real constants and $\det A \neq 0$. Then the unique rest point (equilibrium point) of (LC_{**}) is $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$. Let λ_1 and λ_2 be the eigenvalues of the matrix A. Classify the behavior of the solutions of (LC_{**}) and sketch its corresponding phase portraits.
Hint: Discuss the various cases of λ_1 and λ_2.

5. Consider the following linear periodic system

$$X' = A(t)X, \qquad (LP)$$

where $A(t) = [a_{ij}(t)]_{n\times n} \in \mathbf{R}^{n\times n}$ is continuous on $\mathbf{R}$ and $A(t) = A(t+T)$.

(i) State the Floquet Theorem, the definition of the Floquet multipliers (characteristic multipliers) of (LP).

(ii) Explain that the Floquet multipliers (characteristic multipliers) of (LP) are uniquely determined by the system (LP).

(iii) Let

$$A(t) = \begin{bmatrix} 1 & 1 \\ \\ 0 & \dfrac{\sin t + \cos t}{2 + \sin t - \cos t} \end{bmatrix} = A(t + 2\pi).$$

Compute the Floquet multipliers (characteristic multipliers) of (LP).

Happy Chinese New Year!!