Vector Spaces

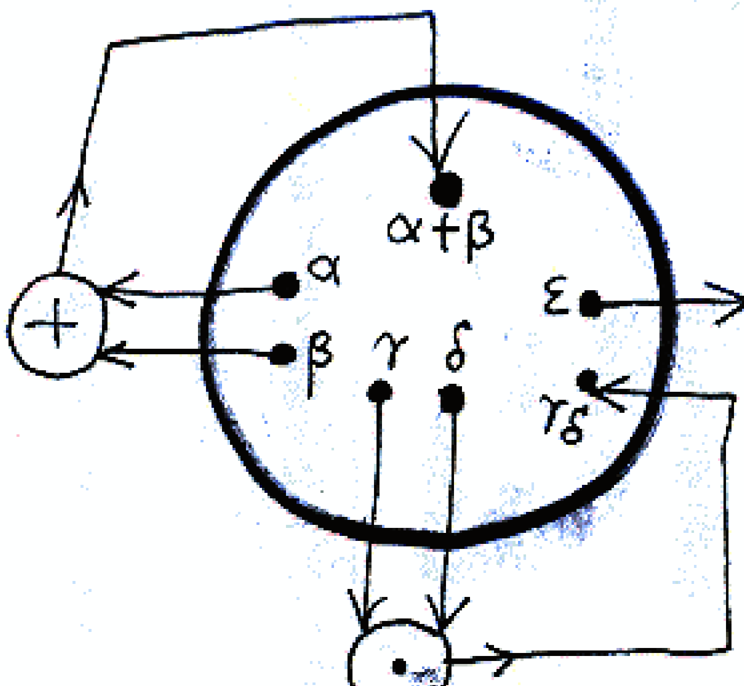
- $(V, F, \oplus, \odot)$

V : a set of objects (usually referred to as vectors)

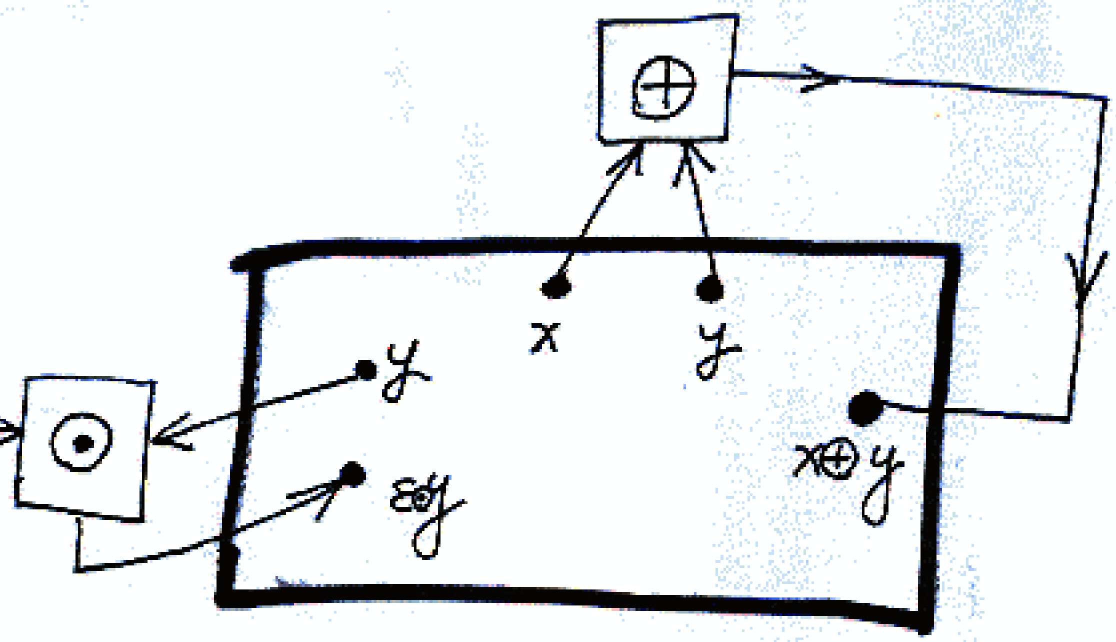
F : a field.

$\oplus$: vector sum

$\odot$: scalar multiplication



Field F



Vector space V

(VS1) $x \oplus y = y \oplus x$

(VS2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

(VS3) $\exists\ 0 \in V$ s.t. $x \oplus 0 = x$ for any $x \in V$

(VS4) For each $x \in V$ there exists $y \in V$ s.t. $x \oplus y = 0$

identity element of F

(VS5) $1 \odot x = x$

(VS6) $(ab) \odot x = a \odot (b \odot x)$

(VS7) $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$

(VS8) $(a+b) \odot x = (a \odot x) \oplus (b \odot x)$

- elements in the field $\mathbb{F}$ are called scalars
- elements in the vector space V are called vectors.
- $\mathbb{F}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \cdots, a_n \in \mathbb{F} \right\}$

entry $\rightarrow$ $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ is called a column vector

$(a_1, a_2, \cdots, a_n)$ is called a row vector

- $$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$ is called an **$m \times n$ matrix.**

  If each **entry** $a_{ij} \in \mathbb{F}$ $(1 \le i \le m,\ 1 \le j \le n)$ then we say $A \in M_{m\times n}(\mathbb{F})$.

- $A = (a_{ij})_{m\times n}$. If $A \in M_{m\times n}(\mathbb{F})$ then $A_{ij} \stackrel{\text{def}}{=\!=} a_{ij}$.
- **square matrix**, **zero matrix**
- We say that $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{r\times s}$ are **equal** if $m = r$, $n = s$ and $a_{ij} = b_{ij}$.

- The ith row of $A=(a_{ij})_{m\times n}$ is $(a_{i1}, a_{i2}, a_{i3}, \cdots, a_{in})$.

  The jth column of $A=(a_{ij})_{m\times n}$ is $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$.

- Matrix addition and scalar multiplication:

  For $A, B \in M_{m\times n}(F)$ and $c \in F$,

  $$(A+B)_{ij} = A_{ij} + B_{ij}, \quad (cA)_{ij} = cA_{ij}$$

- If A_j is the jth column of $A \in M_{m\times n}$ then we write $A = (A_1, A_2, \cdots, A_n)$. If R_i is the ith row of A then we write $A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix}$.

Example of Vector Space

- $\mathcal{F}(S,\mathbb{F}) \overset{\text{def}}{=\!=} \{ f : f : S \longrightarrow \mathbb{F} \}$

  $S \neq \phi$ a field $(\mathbb{F}, \cdot, +)$ function

- $f, g \in \mathcal{F}(S,\mathbb{F})$ are called equal if $f(s) = g(s)$ for any $s \in S$.

- For $f, g \in \mathcal{F}(S,\mathbb{F})$ and $c \in \mathbb{F}$, we define

  $$(f \oplus g) = f(s) + g(s) \quad \text{and} \quad (c \odot f)(s) = c \cdot f(s)$$

  for each $s \in S$.

Fact: $\left(\mathcal{F}(S,\mathbb{F}), \mathbb{F}, \oplus, \odot \right)$ is a vector space.

Example of Vector Space

- $S \overset{\text{def}}{=} \{ (a_1, a_2) : a_1, a_2 \in \mathbb{R} \}$
- For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in \mathbb{R}$ define

$$(a_1, a_2) \oplus (b_1, b_2) = (a_1 + b_1, a_2 + b_2) \quad \text{and}$$

$$c \odot (a_1, a_2) = (ca_1, ca_2) \quad \text{and}$$

$$(a_1, a_2) \triangle (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$$

Fact: $(S, \mathbb{R}, \oplus, \odot)$ is a vector space, but

$(S, \mathbb{R}, \triangle, \odot)$ is NOT a vector space.

Why?

- **polynomial** $f(x)$:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

coefficient of x^n

- **degree** of a polynomial.

- $P(\mathbb{F}) \stackrel{\text{def}}{=}$ the set of all polynomials with coefficients from $\mathbb{F}$.

a field

Fact: $(P(\mathbb{F}), \mathbb{F}, \oplus, \odot)$ is a vector space.

<u>Thm 1.1</u> p11 If $x, y, z \in V$ and $x \oplus z = y \oplus z$ then $x = y$

a vector space $(V, F, \oplus, \odot)$

<u>Pf</u>: $\exists\ \hat{z} \in V$ such that $z \oplus \hat{z} = 0$ $(\because VS4)$

$$x \oplus z = y \oplus z$$

$$\Rightarrow (x \oplus z) \oplus \hat{z} = (y \oplus z) \oplus \hat{z}$$

$$\Rightarrow x \oplus (z \oplus \hat{z}) = y \oplus (z \oplus \hat{z}) \quad (\because VS2)$$

$$\Rightarrow x \oplus 0 = y \oplus 0 \quad (\because VS4)$$

$$\Rightarrow x = y \quad (\because VS3)$$

Corollary 1: The vector 0 described in (VS3) is unique.

pf: Assume $\exists$ a vector $z \in V$ s.t. $x \oplus z = x$ for each vector x in the vector space $(V, F, \oplus, \odot)$.

Then

$$z \underset{\substack{\text{(VS3)}\\ \text{(VS1)}}}{=} 0 \oplus z \underset{\text{hypothesis}}{=} 0$$

QED

Corollary 2: The vector y described in (VS4) is unique.

pf: Assume $\exists$ y and $\hat{y}$ such that $x \oplus y = x \oplus \hat{y} = 0$.

Then $y = y \oplus 0 = y \oplus (x \oplus \hat{y}) = (y \oplus x) \oplus \hat{y} = 0 \oplus \hat{y} = \hat{y}.$

- zero vector of V
- additive inverse of x is denoted by $-x$.

Thm 1.2: In any vector space $(V, F, \oplus, \odot)$ we have

(a) $0 \odot x = \mathbf{0}$ for each $x \in V$.

(b) $(-a) \odot x = -(a \odot x) = a \odot (-x)$ for each $a \in F$, each $x \in V$

(c) $a \odot \mathbf{0} = \mathbf{0}$ for each $a \in F$

Pf: (a) $(0 \odot x) \oplus (0 \odot x) \underset{\uparrow \text{(VS8)}}{=} (0+0) \odot x = 0 \odot x \underset{\uparrow \text{(VS3)}}{=} (0 \odot x) \oplus \mathbf{0}$

(b) $[(-a) \odot x] \oplus (a \odot x) \underset{\uparrow \text{(VS8)}}{=} (-a+a) \odot x = 0 \odot x \underset{\uparrow \text{(a)}}{=} \mathbf{0}$

$[a \odot (-x)] \oplus (a \odot x) \underset{\uparrow \text{(VS7)}}{=} a \odot [(-x) \oplus x] \underset{\uparrow \text{(VS4)}}{=} a \odot \mathbf{0} = ?$

see (c) first!

Pf (c) $a \odot 0$

$\overset{(VS3)}{=} a \odot (0 \oplus 0)$

$= (a \odot 0) \oplus (a \odot 0)$

So we have $a \odot 0 = 0$. why?

- trace$(A) \stackrel{\text{def}}{=\!=} A_{11} + A_{22} + \cdots + A_{nn}$, provided $A \in M_{n\times n}$.

Claim Let $W = \{A \in M_{n\times n}(\mathbb{R}) : \text{trace}(A) = 0\}$.

Then W is a subspace of $M_{n\times n}(\mathbb{R})$.

Fact: Let $W = \{A \in M_{m\times n}(\mathbb{R}) : A_{ij} \geq 0,\ 1 \leq i \leq m,\ 1 \leq j \leq n\}$

Then W is NOT a subspace of $M_{m\times n}(\mathbb{R})$.