Def: A basis β for a vector space V is a l.i. set of vectors in V s.t. $V = \text{span}(\beta)$. The space V is finite-dimensional if it has a finite basis, otherwise it is called infinite-dimensional.

Remark: Note $\text{span}(\phi) \overset{def}{=} \{0\}$ and ϕ is l.i. So the zero vector space $\{0\}$ has a basis ϕ.

Exa2 In F^n, let $e_1 = (1, 0, 0, \cdots 0)$

$e_2 = (0, 1, 0, \cdots 0)$

$\vdots$

$e_n = (0, 0, 0, \cdots, 0, 1)$.

Then $\{e_1, e_2, \cdots, e_n\}$ is a basis for F^n and is called the standard basis of F^n.

Exa3: In $M_{m\times n}(F)$,

let E^{ij} = the matrix whose only nonzero entry is a 1 in the ith row and jth column.

Then $\{E^{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a basis for $M_{m\times n}(F)$.

Thm 1.8 p43 Let $\beta = \{u_1, u_2, \dots, u_n\}$ be a subset of the vector space V over F. Then

β is a basis for $V \Leftrightarrow$ each $v \in V$ can be uniquely expressed as a l.c. of vectors in β.

linear combination

i.e. $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$

for unique $(a_1, a_2, \dots, a_n) \in F^n$.

Thm 1.9 let S be a finite subset of a vector space V. If $V = \text{span}(S)$ then

(1) some subset of S is a basis for V

(2) V has a finite basis.

pf: (Sketch) $S = \phi \Rightarrow$ trivial. Assume $S = \{\alpha_1, \dots, \alpha_l\}$.

$S \leftarrow \{\alpha_1\}$

For $k = 2$ to l

do If $\alpha_k \notin \text{span}(S)$

then $S \leftarrow S \cup \{\alpha_i\}$

Thm 補1 Let V be a vector space with $V=\mathrm{span}(\{\beta_1,\beta_2,\cdots,\beta_m\})$. Then

(1) any l.i. set of vectors in V is finite.

(2) any l.i. set of vectors in V contains no more than m elements.

pf To show if $S=\{\alpha_1,\alpha_2,\cdots,\alpha_n\}\subseteq V$ and $n>m$ then S is l.d.

distinct vectors.

$V=\mathrm{span}(\{\beta_1,\beta_2,\cdots,\beta_m\})$ implies $\exists\, A_{ij}\in F$ s.t.

$$[\beta_1,\beta_2,\cdots,\beta_m]\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn}\end{bmatrix}=[\alpha_1,\alpha_2,\cdots,\alpha_n]$$

不嚴格，考試時不可這樣寫!

Since $n>m$, it is easy to find scalars $x_1,x_2,\cdots,x_n$ not all zero s.t.

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \\ A_{m1} & \cdots & \cdots & A_{mn}\end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}=\begin{bmatrix} 0 \\ \vdots \\ 0\end{bmatrix}.$$ why?

and hence

Hence $x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n=0$

S is a l.d. set. **QED**

Corollary 1 P46 If V is a finite-dimensional vectorspace, then any two bases of V have the same number of vectors.

Def: The dimension $\dim(V)$ of a finite-dimensional vector space V is defined to be the number of elements in a basis for V.

Exa 7, 9, 10, 11, 12

(7) The zero vector space $\{0\}$ has dimension 0.

(9) $\dim(M_{m\times n}(F)) = mn$

(10) $\dim(P_n(F)) = n+1$

(11) $\dim((\mathbb{C},\mathbb{C},+,\cdot)) = 1$ ← a basis $\{1\}$

(12) $\dim((\mathbb{C},\mathbb{R},+,\cdot)) = 2$ ← a basis $\{1, i\}$

Corollary 2 P47 Suppose $\dim V = n$ and $\beta \subseteq V$.

Then

(a) If $V = \mathrm{span}(\beta)$ then $|\beta| \geq n$.

If $V = \mathrm{span}(\beta)$ and $|\beta| = n$ then β is a basis for V.

(b) If β is l.i. and $|\beta| = n$ then β is a basis for V.

(c) If β is l.i. then V has a basis β' such that $\beta \subseteq \beta'$. (i.e. β can be extended to a basis for V)

pf: 補1 $\Rightarrow$ (a)

Corollary 1 P46 + proof of 補1 $\Rightarrow$ (b)

proof of 補1 $\Rightarrow$ (c)

EX 15 P48 $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

Please Examples 13, 14 by yourself!

Thm 1.11[P50] let $\dim V < \infty$ and W is a subspace of V. Then

(a) $\dim W < \infty$

(b) $\dim W \leq \dim V$

(c) if $\dim W = \dim V$ then $V = W$

pf: 補1 $\Rightarrow$ (a) and (b)

Thm 1.7[P39] $\Rightarrow$ (c).

EX17[P50] Let $W = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_5 \end{pmatrix} \in F^5 : a_1 + a_3 + a_5 = 0,\ a_2 = a_4 \right\}$

Then (1) W is a subspace of F^5

(2) $\dim(W) = 3$, Why?

EX18,19 let $W = \{ A \in M_{n\times n}(F) : A \text{ is diagonal} \}$

Let $W' = \{ A \in M_{n\times n}(F) : A = A^t \}$. Then

(1) W, W' are subspace of $M_{n\times n}(F)$

(2) $\dim(W) = n$, $\dim(W') = \frac{1}{2} n(n+1)$

Why?