The Lagrange Interpolation Formula

The Lagrange polynomials w.r.t. $\underbrace{c_0, c_1, \cdots, c_n}_{\text{distinct}}$ scalars in an infinite field.

def $f_0(x), f_1(x), \cdots, f_n(x)$, where

$$f_i(x) \stackrel{def}{=} \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x - c_k)}{(c_i - c_k)}$$

Let $\beta = \{f_0, f_1, \cdots, f_n\} \subseteq P_n(F)$

Note that $f_i(c_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$

Claim: (1) β is a l.i. subset of $P_n(F)$

(2) β is a basis for $P_n(F)$.

(3) If $g \in P_n(F)$ then $g = \sum_{i=0}^{n} g(c_i) f_i$ is the unique representation of g as a l.c. of vectors in β.

pf: (1) If $\sum_{i=0}^{n} a_i f_i = 0$ zero fun.

then $a_j = \sum_{i=0}^{n} a_i f_i(c_j) = 0$, $j = 0, 1, 2, \cdots, n$.

(2) $\dim P_n(F) = |\beta|$ and β is l.i. imply that β is a basis for $P_n(F)$ by Corollary 2 (b)

pf (3) If $g=\sum_{i=0}^{n} a_i f_i$ then

$g(c_j)= \sum_{i=0}^{n} a_i f_i(c_j)=a_j$. Done.

Fact: The polynomial function

$$g=\sum_{i=0}^{n} b_i f_i$$ (Lagrange interpolation formula)

is the unique polynomial in $P_n(F)$ s.t.
$g(c_j)=b_j$

why?

Example: Find the unique polynomial g in $P_3(\mathbb{R})$ such that $g(1)=8$, $g(2)=5$ and $g(3)=-4$.

(Hint) Let $c_0=1$ $b_0=8$
$c_1=2$ $b_1=5$
$c_2=3$ $b_2=-4$

Fact: If $f\in P_n(F)$ and $f(c_i)=0$ for $n+1$ distinct scalars $c_0, c_1, \dots, c_n \in F$, then $f=0$. zero fun.