Def $T: V \to W$ is a linear transformation from V to W if for any $x, y \in V$ and $c \in F$,

vector spaces over F

ⓐ $T(x+y) = T(x) + T(y)$

ⓑ $T(cx) = c\,T(x)$

$\}$ T is linear

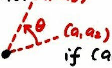
Def: $L(V,W) = \{T : T \text{ is a linear transformation from } V \text{ to } W\}$

Fact: ⓐ $T \in L(V,W) \Rightarrow T(x-y) = T(x) - T(y)$ for all $x, y \in V$.

ⓑ $T \in L(V,W) \Rightarrow T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$

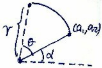
Exa2 P66 Define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. (a_1', a_2')

$$T_\theta(a_1,a_2) = \begin{cases} (a_1', a_2') & \text{where} \\ 0 & \text{if } (a_1,a_2) = (0,0) \end{cases}$$



if $(a_1,a_2) \neq (0,0)$

Then $T_\theta(a_1,a_2) \in L(\mathbb{R}^2, \mathbb{R}^2)$, since



$$T_\theta(a_1,a_2) = (a_1,a_2)\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$a_1' = r\cos(\alpha+\theta)$

$a_2' = r\sin(\alpha+\theta)$

$a_1 = r\cos\alpha$, $a_2 = r\sin\alpha$

補題: Let $T \in L(V,W)$. Then

$T(0_v) = 0_w$

zero vector of V — zero vector of W

pf: $T(0_v) + 0_w$

$= T(0_v)$

$= T(0_v + 0_v)$

$= T(0_v) + T(0_v)$ $(\because T \in L(V,W))$

Cancellation Law for Vector Addition implies $T(0_v) = 0_w$

Def: Let $T \in L(V, W)$.

null space (kernel) $N(T)$ of $T \stackrel{def}{=} \{x \in V : T(x) = \underset{\text{zero vector of } W}{0_W}\}$

range (image) $R(T)$ of $T \stackrel{def}{=} \{T(x) : x \in V\}$

Thm 2.1 P68 Let $T \in L(V, W)$. Then $N(T)$ and $R(T)$ are subspace of V and W respectively.

Pf: By 補充資料1.

Thm 2.2 P68 Let $T \in L(V, W)$. If $\beta = \{v_1, \cdots, v_n\}$ is a basis for V, then

$R(T) = \text{Span}(T(\beta))$, where

$T(\beta) = \{T(v_1), T(v_2), \cdots, T(v_n)\}$

(Hint) $w \in W \Rightarrow w = T(v) = T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i T(v_i)$.

Def: Let $T \in L(V,W)$. If $\dim N(T) < \infty$ and $\dim R(T) < \infty$, we define

$$\text{nullity}(T) \stackrel{\text{def}}{=\!=} \dim(N(T)) \quad \text{nullity of } T$$

$$\text{rank}(T) \stackrel{\text{def}}{=\!=} \dim(R(T)) \quad \text{rank of } T$$

補充 Let $T \in L(V,W)$, $\dim(V) = n$.
If $\alpha = \{v_1, v_2, \cdots, v_k\}$ be a basis for $N(T)$ and $\beta = \{v_1, v_2, \cdots, v_k, v_{k+1}, v_{k+2}, \cdots, v_n\}$ is a basis for V. Then $\{T(v_{k+1}), T(v_{k+2}), \cdots, T(v_n)\}$ is a basis for $R(T)$.

pf: (sketch) $w \in R(T) \Rightarrow \exists\, v \in V$ s.t. $w = T(v)$

Note that

$$\begin{aligned} w = T(v) &= T\left(\textstyle\sum_{i=1}^{n} a_i v_i\right) \\ &= \textstyle\sum_{i=1}^{n} a_i T(v_i) \\ &= \textstyle\sum_{i=k+1}^{n} a_i T(v_i) \end{aligned}$$

$b_{k+1} = b_{k+2} = \cdots = b_n = 0$

If $\sum_{i=k+1}^{n} b_i T(v_i) = 0_W$ then $\sum_{i=k+1}^{n} b_i v_i \in N(T) = \text{span}(\alpha)$ implies

Dimension Theorem

Thm P70 Let $T \in L(V,W)$ and $\dim V < \infty$.

Then $\text{nullity}(T) + \text{rank}(T) = \dim(V)$

Pf: Suppose $\dim(V) = n$, $\dim(N(T)) = k$

and $\{u_1, u_2, \cdots, u_k\}$ is a basis for $N(T)$.

Claim1 $\{u_1, \cdots, u_k\}$ can be extended to a basis

$\beta = \{u_1, \cdots, u_k, u_{k+1}, \cdots, u_n\}$ for V.

Claim2: $\{T(u_{k+1}), \cdots, T(u_n)\}$ is a basis for $R(T)$.

Therefore $\dim(R(T)) = n-k$. QED.

Thm 2.4 P71 Let $T \in L(V,W)$. Then

T is one-to-one $\Leftrightarrow$ $N(T) = \{0\}$

Thm 2.5 P71 Let $T \in L(V,W)$ and $\dim V = \dim W < \infty$.

Then T is 1-to-1 $\Leftrightarrow$ T is onto $\Leftrightarrow$ $\text{rank}(T) = \dim(V)$.

Note: $\dim\{0\} = 0$

pf (Thm 2.5)

Thm 2.4

T is 1-to-1 $\Leftrightarrow N(T)=\{0\}$

$\Leftrightarrow \text{rank}(T)=\dim(V)$

$\Leftrightarrow \dim(R(T))=\dim(W)$ ($\because \dim V = \dim W$)

$\Leftrightarrow R(T)=W$ ($\because$ Thm 1.11 p50)

Thm 2.6 p72 Suppose $\{v_1,\cdots,v_n\}$ is a basis for V and V and W are vector spaces over F. For $w_1, w_2,\cdots,w_n \in W$, $\exists!\ T\in L(V,W)$ such that $T(v_i)=w_i$ for $i=1,2,\cdots,n$.

pf: (sketch) Define $T: V\to W$ s.t. for $x\in V$

$T(x)=\sum_{i=1}^{n} a_i w_i$, where $x=\sum_{i=1}^{n} a_i v_i$

Show ⓐ T is well defined.

ⓑ T is linear ⓒ T is unique.

ⓓ $T(v_i)=w_i\ \forall i$.

Let $U\in L(V,W)$ s.t. $U(v_i)=w_i$

$U(x)=\sum_{i=1}^{n} a_i U(v_i) = \sum_{i=1}^{n} a_i w_i = \sum_{i=1}^{n} a_i T(v_i) = T(x)$.