p79

**Def** If $\dim V < \infty$, an ordered basis for $V$ is a basis for $V$, together with a specific order.

Remark: We shall engage in a slight abuse of notation to describe an ordered basis by saying that $\beta = \{\alpha_1, \cdots, \alpha_n\}$ is an ordered basis for $V$.

**Ex:** see p43

① $\{e_1, e_2, \cdots, e_n\}$ is called the standard ordered basis for $\mathbb{F}^n$

② $\{1, x, \dots, x^n\}$ is called the standard ordered basis for $P_n(F)$.

③ The two ordered basis $\beta, \gamma$ for $\mathbb{F}^3$, where $\beta = \{e_1, e_2, e_3\}$, $\gamma = \{e_1, e_3, e_2\}$, are not the same.

**Def:** Let $\beta = \{\alpha_1, \cdots, \alpha_n\}$ be an ordered basis for $V$. For a $x \in V$, say $x = \sum_{i=1}^n a_i \alpha_i$, we define

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

coordinate vector of $x$ relative to the order basis $\beta$

**Ex:** Let $\beta=\{1, x, x^2\}$ be an ordered basis for $P_2(\mathbb{R})$. Let $f(x)=-7x^2+6x+4$, then

$$[f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$$

Let $\beta'=\{3x+2, 1, x^2\}$ be another ordered basis for $P_2(\mathbb{R})$, then

$$[f]_{\beta'} = \begin{pmatrix} 2 \\ 2 \\ -7 \end{pmatrix}$$

**Def:** Let $\beta=\{\alpha_1, \dots, \alpha_n\}$ be an ordered basis for $V$ and $\beta'=\{\beta_1, \dots, \beta_m\}$ an ordered basis for $W$. Let $T \in L(V, W)$, then for each $j$, $1 \le j \le n$, $T\alpha_j$ is uniquely expressible as a l.c.

$$T\alpha_j = \sum_{i=1}^m A_{ij}\beta_i.$$

The $m \times n$ matrix $A=(A_{ij})$ is called the matrix of $T$ in the ordered bases $\beta$ and $\beta'$, and is denoted by $[T]_\beta^{\beta'}$.

If $\beta=\beta'$ we write $[T]_\beta^{\beta'}$ as $[T]_\beta$

**Homework:** Show that if $U \in L(V,W)$ and $[U]_\beta^{\beta'} = [T]_\beta^{\beta'}$ then $U = T$.

# How to Calculate $[T]_\beta^{\beta'}$

**Ex3** Let $T \in L(\mathbb{R}^2, \mathbb{R}^3)$ s.t.

$$T(a_1, a_2) = (a_1 + 3a_2,\ 0,\ 2a_1 - 4a_2)$$

Let $\beta$ and $\gamma$ be the standard ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$. Let $\gamma' = \{e_3, e_2, e_1\}$.

Try to find $[T]_\beta^\gamma$ and $[T]_\beta^{\gamma'}$.

**Answer:** $[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$, $[T]_\beta^{\gamma'} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}$

**Thm 2.7** Let $T, U \in L(V,W)$. $V, W$ are vector spaces over a field $F$.

(a) $a \in F \Rightarrow aT + U \in L(V,W)$

(b) $(L(V,W), F, +, \cdot)$ is a vector space.

$(T+U)(x) = T(x) + U(x)$

$(aT)(x) = aT(x)$.

**Notation:** $\mathcal{L}(V,W) \overset{\text{def}}{=\!=} L(V,W)$

$\mathcal{L}(V) \overset{\text{def}}{=\!=} \mathcal{L}(V,V)$

**Thm 2.8** P83 Let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ be ordered bases for $V$ and $W$ respectively. Let $T, U \in \mathcal{L}(V,W)$. Then

ⓐ $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$ and

ⓑ $[aT]_\beta^\gamma = a[T]_\beta^\gamma$

**Pf** (sketch) $T(v_j) = \sum_{i=1}^m a_i w_i$

$\Longrightarrow (aT)(v_j) = \sum_{i=1}^m a \cdot a_i w_i$

**Ex 5** P83 Please see this example by yourself.