Thm 2.9 Let V, W, and Z be vector spaces over the same field F. Let $T \in \mathcal{L}(V, W)$ and $U \in \mathcal{L}(W, Z)$. Then $UT \in \mathcal{L}(V, Z)$.

Thm 2.10 Let $T, U_1, U_2 \in \mathcal{L}(V)$. Then

ⓐ $T(U_1 + U_2) = TU_1 + TU_2$ and
$(U_1 + U_2)T = U_1T + U_2T$

ⓑ $T(U_1U_2) = (TU_1)U_2$

ⓒ $TI = IT = T$

ⓓ $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$

Remark ① The product of $m \times n$ matrix A and $n \times p$ matrix B.

② It is not true that $AB = BA$.

③ $(AB)^t = B^t A^t$

Thm 2.11 Let α, β, and γ are ordered bases for V, W and Z respectively. Let $T \in \mathcal{L}(V, W)$ and $U \in \mathcal{L}(W, Z)$. Then

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta .$$

Ex2 p89 Let $U \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$
$T \in \mathcal{L}(P_2(\mathbb{R}), P_3(\mathbb{R}))$ s.t.

$$U(f(x)) = f'(x) \quad \text{and} \quad T(f(x)) = \int_0^x f(t)\,dt.$$

Let α and β be the standard ordered bases for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively.

Then

$$[UT]_\beta = [U]_\alpha^\beta [T]_\beta^\alpha$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = [I]_\beta$$

Notation ① Kronecker delta $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

② $n \times n$ identity matrix $I_n = (\delta_{ij})$

③ O is the zero matrix.

Thm 2.12 See this Thm by yourself.

$A^n \overset{\text{def}}{=} \underbrace{A \cdot A \cdot \cdots \cdot A}_{n \text{ times}}$ where $A \in M_{n\times n}$.

Quiz: Is it true that $A^2 = 0$ implies $A = 0$?

2x2 zero matrix.

Thm 2.13: Let $B = (B_{\cdot 1}, B_{\cdot 2}, \cdots, B_{\cdot p}) \in M_{n\times p}$, $A \in M_{m\times n}$ and $AB = ((AB)_{\cdot 1}, (AB)_{\cdot 2}, \cdots, (AB)_{\cdot p})$.

Then (a) $(AB)_{\cdot j} = A B_{\cdot j}$

(b) $B_{\cdot j} = B \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← j-th component

vector spaces over field F

Thm 2.14 p91 Let $T \in \mathcal{L}(V, W)$, $\dim V < \infty$, $\dim W < \infty$. Let β and γ be ordered basis for V and W respectively. Then for each $u \in V$, we have

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$$

重要!

pf: Consider F as a vector space over F.

Let $\alpha = \{1\}$ be an order basis for F.

$$\begin{tikzcd} F^{\alpha=\{1\}} \arrow[r, "f"] \arrow[dr, "Tf"'] & V^{\beta} \arrow[d, "T"] \\ & W_{\gamma} \end{tikzcd}$$

Fix $u \in V$, define $f \in \mathcal{L}(F, V)$ s.t. $f(a) = au$. Then $[T(u)]_\gamma = [Tf(1)]_\gamma$

$= [Tf]_\alpha^\gamma \underset{\text{Thm 2.11}}{=} [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f(1)]_\beta = \text{RHS}$

Ex3 p91 Let $T \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$ s.t. $T(f) = f'$. Let β and γ be standard ordered basis for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ resp.

Let $p(x) = 2 + x^2 + x^3 \in P_3(\mathbb{R})$.

Then $[T(p(x))]_\gamma = [2x + 3x^2]_\gamma = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\|$$

$$[T]_\beta^\gamma [p(x)]_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

Def: Let $A \in M_{m \times n}(F)$. The mapping $L_A : F^n \to F^m$ is defined by $L_A(x) = Ax$ for each column vector $x \in F^n$.

L_A is called a left-multiplication transformation.

Thm 2.15 p93 please see (b) (c) (d), (f) by yourself.

Thm 2.15 p93 Let $A \in M_{m\times n}(F)$. Let β and γ be the standard ordered bases for F^n and F^m.

Then $L_A \in \mathcal{L}(F^n, F^m)$, moreover we have

(a) $[L_A]_\beta^\gamma = A$

(e) If $E \in M_{n\times p}(F)$ then $L_{AE} = L_A L_E$.

指 L_A 和 L_E 两函数合成!

Pf: (a) Let $I_n = [e_1, e_2, \cdots, e_n]$.

$L_A(e_j) = Ae_j$ = the jth column of A.

Therefore $[L_A]_\beta^\gamma = A$.

(b) Let $I_p = [e_1, e_2, \ldots, e_p]$.

Note that L_{AE} and $L_A L_E \in \mathcal{L}(F^p, F^m)$.

$$(L_{AE})(e_j) = (AE)e_j = A(Ee_j) = A\,L_E(e_j)$$

$$= L_A(L_E(e_j)).$$

Thm 2.16 p93: Mathix multiplication is associative.

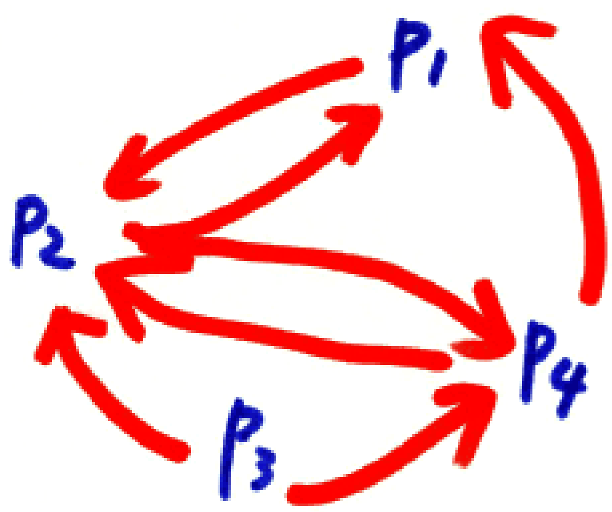
Please see Thm 2.16 by yourself!

Applications

(A) Four people P_1, P_2, P_3 and P_4.

Let $A_{ij} = \begin{cases} 1 & \text{if } i \text{ can send a message to } j \\ 0 & \text{o.w.} \end{cases}$

$\vec{G}$



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

A is called the incidence matrix of the digraph $\vec{G}$.

Fact: $(A+A^2+\cdots+A^m)_{ij}$ is the number of ways in which i can send a message to j in at most m stages.

pf (sketch)

$$(A^2)_{ij} = \sum_{k=1}^{3} A_{ik}A_{kj}$$ = the # of way that i can send a message to j in exactly 2 stages.

$$(A^3)_{ij} = \sum_{k=1}^{3} (A^2)_{ik}A_{kj}$$ = the # of ways that i can send to j in exactly 3 stages.