2.4 Invertibility & Isomorphisms

Def: p99 $T \in \mathcal{L}(V, W)$. A fun. $U: W \to V$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$.
If T has an inverse then T is called invertible.

Note: If T is invertible, then the inverse of T is unique and is denoted by T^{-1} (see p55-2)

Fact If T and U are invertible funs.
then ① $(TU)^{-1} = U^{-1}T^{-1}$
② T^{-1} is invertible and $(T^{-1})^{-1} = T$.
③ $T^{-1}(0_W) = 0_V$ $\quad$ $(\because T^{-1}T(0_V) = 0_V \Rightarrow T^{-1}(0_W) = 0_V)$

Thm 2.5 (revisited) p100 let $T \in \mathcal{L}(V, W)$ and $\dim V = \dim W < \infty$.
Then T is invertible $\iff$ $\operatorname{rank}(T) = \dim V$

pf: (do not use dimension Thm).
Let $\beta = \{v_1, v_2, \cdots, v_n\}$ be a basis for V.
Claim $\alpha = \{Tv_1, Tv_2, \cdots, Tv_n\}$ is a basis for $R(T)$
pf: $w \in R(T) \Rightarrow \exists\, v \in V$ s.t. $w = T(v)$
$\Rightarrow w \in \operatorname{span}(\alpha)$. Thus $R(T) = \operatorname{span}(\alpha)$.

$c_1Tv_1 + c_2Tv_2 + \cdots + c_nTv_n = 0_W$
$\Rightarrow T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = 0_W$
$\Rightarrow T^{-1}(T(c_1v_1 + c_2v_2 + \cdots + c_nv_n)) = T^{-1}(0_W) = 0_V$
$\Rightarrow c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0_V \Rightarrow c_1 = c_2 = \cdots = c_n = 0$ QED

Appendix B

A function $f: A \to B$ is called **invertible** if $\exists$ a function $g: B \to A$ s.t.

① $(f \circ g)(y) = y$ for all $y \in B$

② $(g \circ f)(x) = x$ for all $x \in A$.

Fact: If such a fun. g exists, then it is **unique** and is called the **inverse of f**, denoted by f^{-1}.

Fact: f is invertible $\Longleftrightarrow$ f is **1-to-1 and onto**.

Thm 2.17 p100 $T \in \mathcal{L}(V, W)$ is invertible

$\Longrightarrow T^{-1} \in \mathcal{L}(W, V)$

pf: $T^{-1}(cw_1 + w_2) = T^{-1}(cT(v_1) + T(v_2))$

$= cv_1 + v_2 = cT^{-1}(w_1) + T^{-1}(w_2)$

Def: $A \in M_{n \times n}(\mathbb{R})$. A is called invertible if $\exists\, B \in M_{n \times n}$ s.t. $AB = BA = I_n$.

Fact: If such B exists, then it is unique and is called the inverse of A and is denoted by A^{-1}.

Lemma p101 $T \in \mathcal{L}(V, W)$ is invertible. Then

① $\dim V < \infty \Leftrightarrow \dim W < \infty$

② $\dim V < \infty \Longrightarrow \dim V = \dim W$

pf: ① T is onto $\Rightarrow W \subseteq R(T) \Rightarrow \dim W \leq \dim R(T) = \operatorname{rank} T$
$= \dim V$

Thm 2.5 p71

② T^{-1} is onto $\Rightarrow \dim W \geq \dim V$ by ①

Thm 2.18 p101 $T \in \mathcal{L}(V, W)$, $\dim V < \infty$, $\dim W < \infty$. V, W have ordered bases β, γ respectively.

Then

① T is invertible $\Longleftrightarrow [T]_\beta^\gamma$ is invertible

② T^{-1} exists $\Longrightarrow [T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$

Pf (of Thm 2.18) Let $\dim V = n$.

T is invertible $\Rightarrow \dim V = \dim W = n$.

$$T \text{ is invertible} \Rightarrow I_n = [I]_\beta^\beta \quad ([I]_\gamma^\gamma)$$

$$= [T^{-1}T]_\beta^\beta \quad ([TT^{-1}]_\gamma^\gamma)$$

$$= [T^{-1}]_\gamma^\beta [T]_\beta^\gamma \quad ([T]_\beta^\gamma [T^{-1}]_\gamma^\beta)$$

$\Rightarrow [T]_\beta^\gamma$ is invertible.

and $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$.

$[T]_\beta^\gamma$ is invertible

$\Rightarrow \exists B \in M_{n \times n}$ s.t. $[T]_\beta^\gamma B = B [T]_\beta^\gamma = I_n$

Define $U \in \mathcal{L}(W, V)$ s.t. $[U(w)]_\beta = B [w]_\gamma$ for $\forall w \in W$.

We have $[(UT)(v)]_\beta = [U(T(v))]_\beta$

for any $v \in V$

$$= B [T(v)]_\beta \quad (\text{by definition of } U)$$

$$= B [T]_\beta^\gamma [v]_\beta \quad (\text{by Thm 2.14 } ^{p91})$$

hypothesis

$$= I_n [v]_\beta = [v]_\beta$$

So $UT = I_V$ and similarly, $TU = I_W$.

Def: Let V, W be vector spaces over the field F. We say that V is isomorphic to W if $\exists\ T \in \mathcal{L}(V,W)$ that is invertible. And such T is called an isomorphism from V onto W.

Note: V is isomorphic to $W \Leftrightarrow W$ is isomorphic to V

So we use $V \cong W$ to denote that V and W are isomorphic.

Thm 2.19 Let V and W be finite-dimensional vector spaces over the same field. Then

$$V \cong W \iff \dim(V) = \dim(W)$$

pf: $(\Rightarrow)$ $V \cong W \Rightarrow \exists$ invertible $T \in \mathcal{L}(V,W)$

$\Rightarrow \dim V = \dim W$ (by Lemma p101)

$(\Leftarrow)$ Let $\dim V = \dim W = n$.

Let $\beta = \{\beta_1, \beta_2, \cdots, \beta_n\}$, $\gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$

be ordered bases of V and W respectively.

Thm 2.6 p72 $\Rightarrow \exists\ T \in \mathcal{L}(V,W)$ s.t. $T(\beta_i) = \gamma_i$, $i = 1, 2, \cdots, n$

$$\Rightarrow \operatorname{rank}(T) = \dim R(T) = \dim(\operatorname{Span}(T(\beta))) = |\gamma| = \dim V$$

$\Rightarrow T$ is invertible by Thm 2.5 p71 (revisited) p100

So $V \cong W$.

Corollary: let V be a vector space over F.
Then $V \cong F^n \Leftrightarrow \dim V = n$

$\dim V = n$ $\dim W = m$

Thm 2.20 P103 let V, W be vector spaces over F
having ordered bases β and γ resp. The the
fun. $\varphi : \mathcal{L}(V,W) \to M_{m\times n}(F)$, defined by
$\varphi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V,W)$ is an isomorphism.

pf: Claim: $\varphi(cT+U) = c\varphi(T) + \varphi(U)$ for $c \in F$, $T, U \in \mathcal{L}(V,W)$

by Thm 2.8 P82

For $A \in M_{m\times n}(F)$, by Thm 2.6 P72, $\exists!\ T_A \in \mathcal{L}(V,W)$
s.t. $[T_A]_\beta^\gamma = A$, i.e. $\varphi(T_A) = A$.
So φ is one-to-one and onto i.e. φ is
invertible.

QED.

Corollary: $\dim V = n$, $\dim W = m \Rightarrow \dim(\mathcal{L}(V,W)) = mn$

Def: $\dim V = n$ and V has an ordered basis β.
The fun $\varphi_\beta : V \to F^n$ defined by $\varphi_\beta(x) = [x]_\beta$ for $\forall x \in V$
is called the standard representation of V with
respect to β

Thm 2.21 p104 $\dim V < \infty$ and V has ordered basis β
$\Rightarrow$ the standard representation ϕ_β of V w.r.t. β is an isomorphism.

Def: $\dim V = n$, V is a vector space over field F having ordered basis β. The standard representation of V with respect to β is a fun. $\phi_\beta : V \to F^n$ s.t. $\phi_\beta(v) = [v]_\beta$

Ex6 R^2 have two ordered bases

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad \gamma = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}.$$

$$\phi_\beta\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\phi_\gamma\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

Thm 2.21 p104 If V has an ordered basis β and $\dim V < \infty$ then ϕ_β is an isomorphism from V ont F^n.

Ex 7 p105

Define $T \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$ s.t.

$T(f(x)) = f'(x)$.

Let β and γ be the standard ordered bases for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ resp.

Let $\phi_\beta : P_3(\mathbb{R}) \longrightarrow \mathbb{R}^4$

$\phi_\gamma : P_2(\mathbb{R}) \longrightarrow \mathbb{R}^3$

be the corresponding standard representations of $P_3(\mathbb{R})$, $P_4(\mathbb{R})$ resp.

Then

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$