2.5 The Change of coordinate matrix

EX1 P112 Let $\beta = \{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix} \}$, $\beta' = \{ \begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix} \}$

be two ordered bases of $\mathbb{R}^2$.

we have

$$\left[\begin{pmatrix}2\\4\end{pmatrix}\right]_\beta = \begin{pmatrix}3\\-1\end{pmatrix} \quad \text{why?}$$

$$\left[\begin{pmatrix}2\\4\end{pmatrix}\right]_{\beta'} = \begin{pmatrix}1\\0\end{pmatrix}$$

Let $I : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity fun.

Note that $[I]_\beta^{\beta'} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix}$,

Since

$$\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2\\4\end{pmatrix} + \frac{1}{5}\begin{pmatrix}3\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\-1\end{pmatrix} = -\frac{2}{5}\begin{pmatrix}2\\4\end{pmatrix} + \frac{3}{5}\begin{pmatrix}3\\1\end{pmatrix}.$$

We have

$$[I]_\beta^{\beta'} \left[\begin{pmatrix}2\\4\end{pmatrix}\right]_\beta = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix}3\\-1\end{bmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = \left[\begin{pmatrix}2\\4\end{pmatrix}\right]_\beta$$

Thm 2.22 $\dim V < \infty$. Let β and β' be two ordered bases of V. Then

ⓐ $[I_V]_\beta^{\beta'}$ is invertible.

ⓑ $[v]_{\beta'} = [I_V]_\beta^{\beta'}[v]_\beta$ for any $v \in V$.

pf: ① $I_V \in \mathcal{L}(V, V)$ is invertible

$\Rightarrow [I_V]_\beta^{\beta'}$ is invertible ($\because$ p101, Thm 2.18)

②

(P.91) Thm 2.14 $\Rightarrow [v]_{\beta'} = [I_V(v)]_{\beta'}$

$$= [I_V]_\beta^{\beta'}[v]_\beta$$

Def: In above Thm, we say that $[I_V]_\beta^{\beta'}$ changes β-coordinates into β'-coordinates, and $[I_V]_\beta^{\beta'}$ is called a change of coordinate matrix.

p112

Ex1 (revisited)

Note that

$$\left([I]_{\beta}^{\beta'}\right)^{-1} = [I^{-1}]_{\beta'}^{\beta} \quad (\because \text{p101, Thm 2.15})$$

$$= \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

Since

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

And

$$\left([I]_{\beta}^{\beta'}\right)^{-1} \left[\begin{pmatrix} 2 \\ 4 \end{pmatrix}\right]_{\beta'}$$

$$= \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \left[\begin{pmatrix} 2 \\ 4 \end{pmatrix}\right]_{\beta}.$$

Fact: Q changes β-coordinates to β'-coordinates

$\Leftrightarrow$ Q^{-1} changes β'-coordinates to β-coordinate

Ex2: let T be linear operator on $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a-b \\ a+3b \end{pmatrix}.$$

let $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$, $\beta' = \left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ be two ordered bases in $\mathbb{R}^2$.

Note that

$$[T]_\beta = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

note: $[T]_\beta \overset{def}{=} [T]_\beta^\beta$

$$[T]_{\beta'} = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}$$

Moreover

$$\left([I]_\beta^{\beta'}\right)^{-1} [T]_{\beta'} [I]_\beta^{\beta'}$$

$$= \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix} \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = [T]_\beta$$

Thm 2.23 $T \in \mathcal{L}(V)$, $\dim V < \infty$.

V has ordered bases β and β'.

Suppose $Q = [I]_{\beta}^{\beta'}$ (i.e. the change of coordinate matrix that changes β-coordinates into β'-coordinates)

Then $$[T]_{\beta} = Q^{-1}[T]_{\beta'}Q$$

pf: Thm 2.11 says that

$$\begin{aligned}
[T]_{\beta} &= [I^{-1}TI]_{\beta}^{\beta} \\
&= [I^{-1}T]_{\beta'}^{\beta}[I]_{\beta}^{\beta'} \\
&= [I^{-1}]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}[I]_{\beta}^{\beta'} \\
&= \left([I]_{\beta}^{\beta'}\right)^{-1}[T]_{\beta'}^{\beta'}[I]_{\beta}^{\beta'} \quad (\because \text{p101 Thm 2.18}) \\
&= Q^{-1}[T]_{\beta'}Q
\end{aligned}$$

Corollary P115 $A \in M_{n\times n}(F)$ & $\gamma = \{\gamma_1, \dots, \gamma_n\}$ is an ordered basis for F^n. Then

$$[L_A]_\gamma = Q^{-1} A Q$$

where $Q = [\gamma_1, \cdots, \gamma_n]$.

pf: (建議自己再証一次, 不要用 Thm 2.23) Let $\beta = \{e_1, \dots, e_n\}$ be the standard ordered basis for F^n.

Note that $L_A : F^n \to F^n$ s.t. $L_A(x) = Ax$.

So

$$\begin{aligned}
[L_A]_\gamma &= [I^{-1} L_A I]_\gamma \\
&= [I^{-1} L_A]_\beta^\gamma \, [I]_\gamma^\beta \\
&= [I^{-1}]_\beta^\gamma \, [L_A]_\beta^\beta \, [I]_\gamma^\beta \\
&= \left([I]_\gamma^\beta\right)^{-1} [L_A]_\beta^\beta \, [I]_\gamma^\beta \\
&= Q^{-1} [Ae_1, Ae_2, \dots, Ae_n] Q \\
&= Q^{-1} A Q.
\end{aligned}$$

Question: let $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix}$.

let $\gamma = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

① Find $[L_A]_\gamma$.

② let $Q = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Find Q^{-1}.

③ Show $Q^{-1} A Q = [L_A]_\gamma$.

Sol: ① $[L_A]_\gamma = \begin{pmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{pmatrix}$

② let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ be the standard order basis for $\mathbb{R}^3$.

Note that $Q = [I]_\gamma^\beta$.

$$Q^{-1} = [I^{-1}]_\beta^\gamma$$

$$= \begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Def: Let $A, B \in M_{n\times n}(F)$.

We say that B is similar to A if $\exists$ an invertible matrix Q such that

$$B = Q^{-1}AQ$$