

# 3.1 Elementary Matrix Operations and elementary matrices

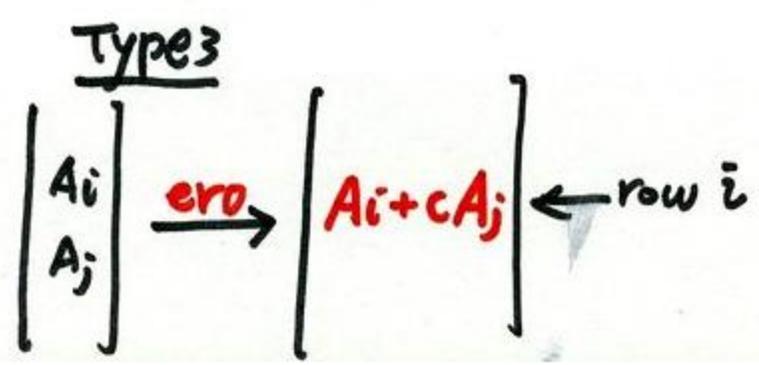
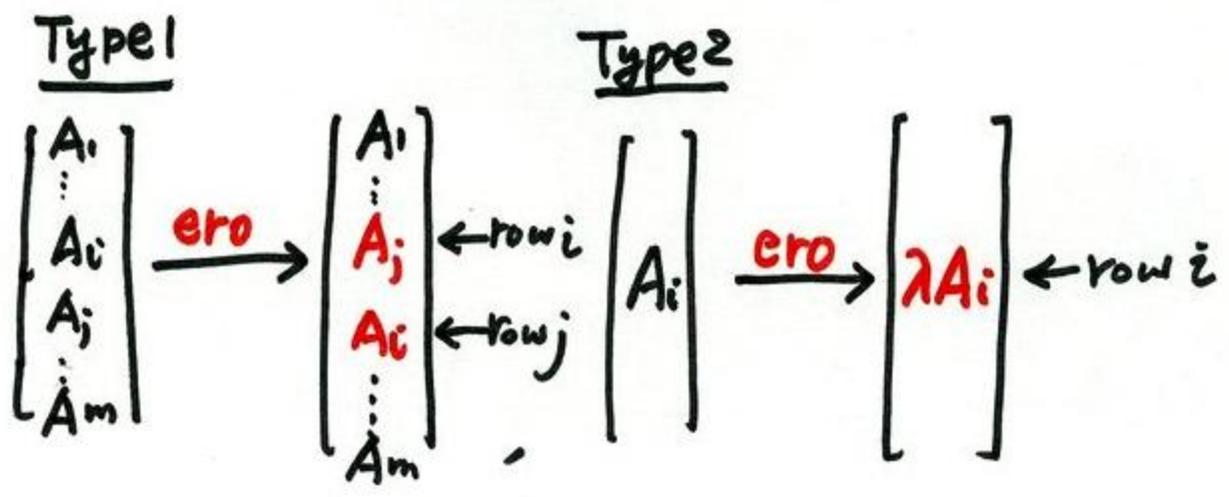
**Def:** Elementary row (column) operations:

Let  $A \in M_{m \times n}$ , say  $A = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$

type 1: interchanging any two rows of  $A$ .

type 2: Replace row  $i$  of  $A$  by itself plus the scalar  $c$  times row  $j$  (where  $i \neq j$ )

type 3: Multiply row  $i$  of  $A$  by the non-zero scalar  $\lambda$



Def: An  $n \times n$  elementary matrix is a matrix obtained by performing an elementary operation on  $I_n$ .

Ex: let  $I_3 = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = [e_1, e_2, e_3]$ .

$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  is an e.m. since

$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} e_1 - 2e_3 \\ e_2 \\ e_3 \end{bmatrix}$  elementary matrix of type 3

$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  is an e.m. since

$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [e_2, e_1, e_3]$  elementary matrix of type 1.

Thm 3.1 let  $A \in M_{m \times n}(F)$ .

① If  $A \xrightarrow{\text{ero}} B$  then  $\exists$  an e.m.  $E \in M_{m \times m}$  s.t.  $B = EA$ .

② If  $A \xrightarrow{\text{eco}} B$  then  $\exists$  an e.m.  $E \in M_{n \times n}$  s.t.  $B = AE$ .

Remark: eco means elementary column operation.

EX: 如何找 Thm 3.1 中的  $E$  呢?

$$\textcircled{1} \begin{matrix} A \rightarrow \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix} \end{matrix} \xrightarrow{\text{ero}(1)} \begin{matrix} \leftarrow \text{type 1} \\ \begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 \end{pmatrix} \end{matrix} \leftarrow B$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{ero}(1)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow E_r$$

Note that  $E_r A = B$

$$\textcircled{2} \begin{matrix} A \rightarrow \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix} \end{matrix} \xrightarrow{\text{eco}(2)} \begin{matrix} \leftarrow C \\ \begin{pmatrix} 1 & 6 & 3 & 4 \\ 2 & 3 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix} \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{eco}(2)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow E_c$$

Note that  $A E_c = C$

Thm 3.2 <sup>p150</sup> Elementary matrices are invertible.

pf: let  $I_n = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$ .

Let  $E_1 = \begin{pmatrix} e_j \\ \leftarrow \text{row } i \\ e_i \\ \leftarrow \text{row } j \end{pmatrix}$ ,  $E_2 = \begin{pmatrix} \lambda e_i \\ \leftarrow \text{row } i \\ \lambda \neq 0 \end{pmatrix}$ ,  $E_3 = \begin{pmatrix} e_i + c e_j \\ \leftarrow \text{row } i \end{pmatrix}$

let  $\tilde{E}_2 = \begin{pmatrix} \frac{1}{\lambda} e_i \\ \leftarrow \text{row } i \end{pmatrix}$ ,  $\tilde{E}_3 = \begin{pmatrix} e_i - c e_j \\ \leftarrow \text{row } i \end{pmatrix}$

Then  $E_1 E_1 = I_n$ ,  $E_2 \tilde{E}_2 = \tilde{E}_2 E_2 = I_n$ ,  $E_3 \tilde{E}_3 = \tilde{E}_3 E_3 = I_n$