3.1 Elementary Matrix Operations and elementary matrices

Def: Elementary row (column) operations:

let $A \in M_{m\times n}$, say $A = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$

<u>type 1</u>: interchangin any two rows of A.

<u>type2</u>: Replace row i of A by itself plus the scalar c times row j (where $i \neq j$)

<u>type2</u>: Multiply row i of A by the <u>non-zero</u> scalar λ

<u>Type1</u>

$$\begin{bmatrix} A_1 \\ \vdots \\ A_i \\ A_j \\ \vdots \\ A_m \end{bmatrix} \xrightarrow{\text{ero}} \begin{array}{c} \begin{bmatrix} A_1 \\ \vdots \\ A_j \\ A_i \\ \vdots \\ A_m \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow \text{row } i \\ \leftarrow \text{row } j \\ \\ \\ \end{array}$$

<u>Type2</u>

$$\begin{bmatrix} \\ A_i \\ \\ \end{bmatrix} \xrightarrow{\text{ero}} \begin{bmatrix} \\ \lambda A_i \\ \\ \end{bmatrix} \leftarrow \text{row } i$$

<u>Type3</u>

$$\begin{bmatrix} A_i \\ A_j \end{bmatrix} \xrightarrow{\text{ero}} \begin{bmatrix} A_i + cA_j \end{bmatrix} \leftarrow \text{row } i$$

Def: An $n \times n$ elementary matrix (e.m. for short) is a matrix obtained by performing an elementary operation on I_n.

Ex: let $I_3 = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = [e_1', e_2', e_3']$.

$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an e.m. since

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} e_1 - 2e_3 \\ e_2 \\ e_3 \end{bmatrix}$$ elementary matrix of type 3

$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an e.m. since

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [e_2', e_1', e_3']$$ elementary matrix of type 1.

Thm 3.1 Let $A \in M_{m \times n}(F)$.

① If $A \xrightarrow{\text{ero}} B$ then $\exists$ an e.m. $E \in M_{m \times m}$ s.t. $B = EA$.

② If $A \xrightarrow{\text{eco}} B$ then $\exists$ an e.m. $E \in M_{n \times n}$ s.t. $B = AE$.

Remark: eco means elementary column operation.

Ex: 如何找 Thm 3.1 中的 E 呢?

① $$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{\overset{\text{type 1}}{\text{ero}(1)}} \begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 \end{pmatrix} \leftarrow B$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{ero}(1)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow E_r$$

Note that $E_r A = B$

② $$A \xrightarrow{\text{eco}(2)} \begin{pmatrix} 1 & 6 & 3 & 4 \\ 2 & 3 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix} \leftarrow C$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{eco}(2)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow E_c$$

Note that $A E_c = C$

Thm 3.2 (P150)

Elementary matrices are invertible.

pf: let $I_n = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$.

Let $E_1 = \begin{pmatrix} e_j \\ e_i \end{pmatrix} \begin{matrix} \leftarrow \text{row } i \\ \leftarrow \text{row } j \end{matrix}$, $E_2 = \begin{pmatrix} \lambda e_i \end{pmatrix} \leftarrow \text{row } i$, $\lambda \neq 0$, $E_3 = \begin{pmatrix} e_i + c e_j \end{pmatrix} \leftarrow \text{row } i$

let $\tilde{E}_2 = \begin{pmatrix} \frac{1}{\lambda} e_i \end{pmatrix} \leftarrow \text{row } i$, $\tilde{E}_3 = \begin{pmatrix} e_i - c e_j \end{pmatrix} \leftarrow \text{row } i$

Then $E_1 E_1 = I_n$, $E_2 \tilde{E}_2 = \tilde{E}_2 E_2 = I_n$, $E_3 \tilde{E}_3 = \tilde{E}_3 E_3 = I_n$